STEP 2003, Paper 2, Q10 - Solution (5 pages; 7/4/24)

[3 types of method that could, in general, be used to create equations for this type of question are:

(a) Creating forces diagrams and applying N2L

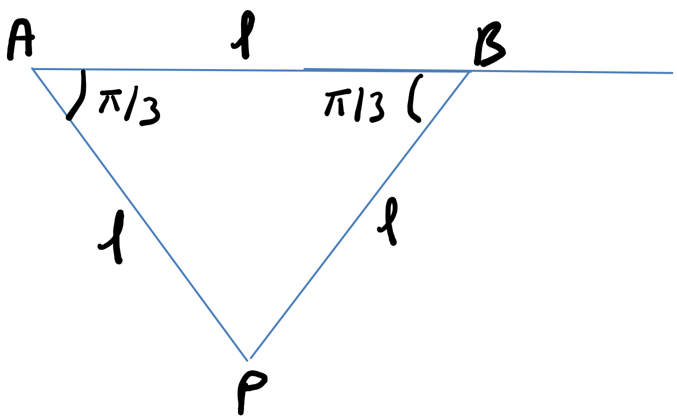
(b) Energy method

(c) Geometrical constraint]

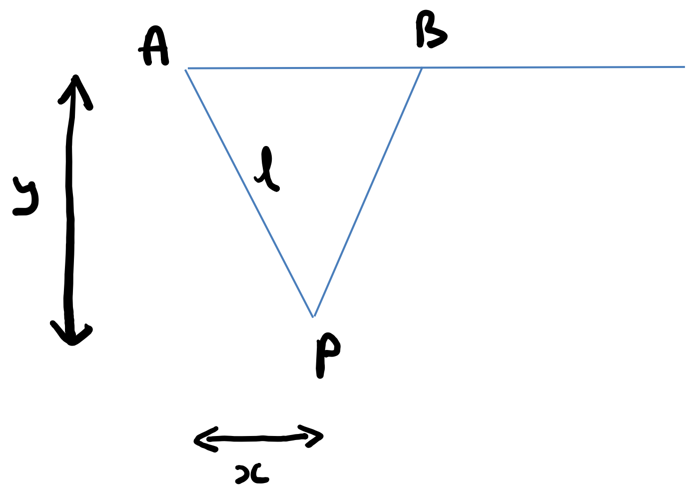
1st Part

[Here (at least in the 1st part of the question), there are unknown forces which prevent the use of methods (a) & (b) (in the case of (b), work will be done by some of the forces).]

Initial position:



Position at a general time:



Then $x^2 + y^2 = l^2$

Differentiating both sides then gives $2x\dot{x} + 2y\dot{y} = 0$

or $x\dot{x} + y\dot{y} = 0$

[where $\dot{x} \equiv \frac{dx}{dt}$]

And differentiating again gives $\dot{x}^2 + x\ddot{x} + \dot{y}^2 + y\ddot{y} = 0$ (*)

Now, the horizontal and vertical displacements of P are

$\frac{l}{2} - x$ (to the left) and $y - lsin(\frac{\pi}{3})$ (downwards) (**)

If a_x and a_y are the horizontal and vertical displacements of P (at a general time), then differentiating (**) twice,

$a_x = -\ddot{x}$ and $a_y = \ddot{y}$

Initially the system is at rest, and so $\dot{x} = \dot{y} = 0$;

and $a_x = a_1$; $a_y = a_2$

Also, initially, $x = \frac{l}{2}$ and $y = (\frac{l}{2})tan\left(\frac{\pi}{3}\right) = \frac{l\sqrt{3}}{2}$

Then, from (*) (initially),

$0 + (\frac{l}{2})(-a_1) + 0 + (\frac{l\sqrt{3}}{2})a_2 = 0,$

so that $a_1 = \sqrt{3}a_2$, as required.

2nd Part

The (horizontal) displacement of B (at a general time) is

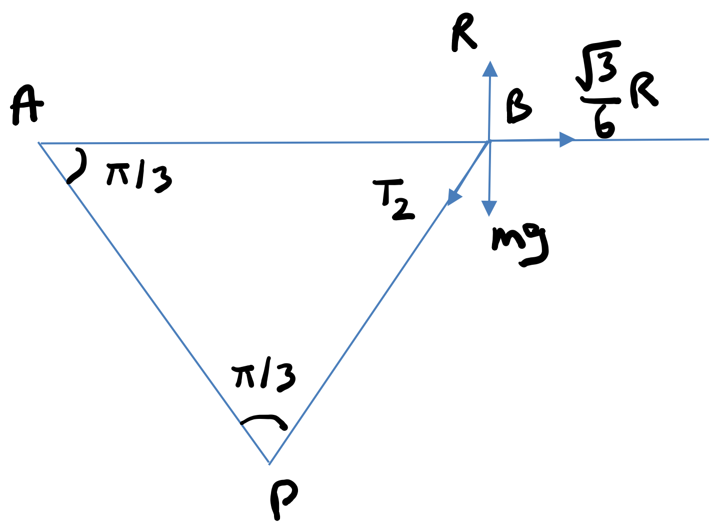
$l - 2x$ (to the left).

Differentiating twice, the acceleration of B is

$-2\ddot{x} = -2(-a_x) = 2a_x$, and initially this equals $2a_1$

3rd Part

Force diagram for B (in the initial position):



Applying N2L:

Vert: $R = mg + T_2 cos(\frac{\pi}{6})$

Horiz: $T_2 cos\left(\frac{\pi}{3}\right) - \frac{\sqrt{3}}{6}R = m(2a_1)$

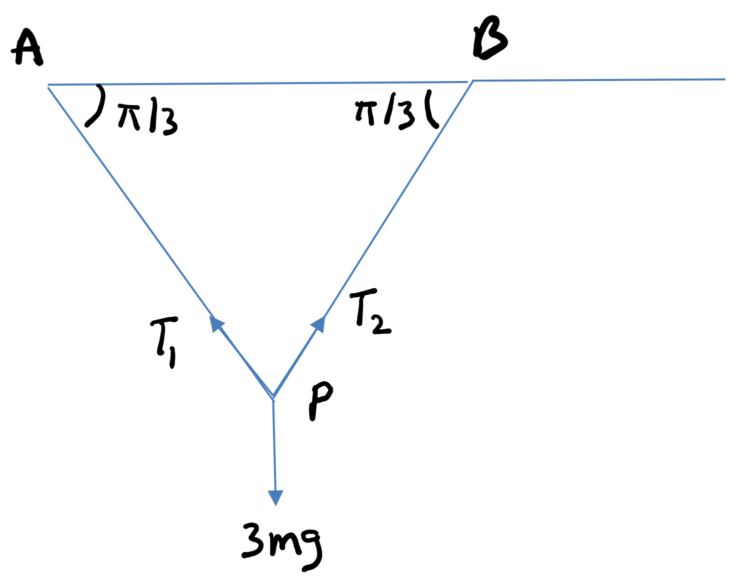
Eliminating R,

$T_2 cos\left(\frac{\pi}{3}\right) - \frac{\sqrt{3}}{6}(mg + T_2 cos\left(\frac{\pi}{6}\right)) = m(2a_1)$

so that $T_2\left(\frac{1}{2} - \frac{\sqrt{3}}{6} \cdot \frac{\sqrt{3}}{2}\right) = m(2a_1 + \frac{\sqrt{3}}{6}g)$,

and $T_2 = \frac{m(2a_1 + \frac{\sqrt{3}}{6}g)}{\frac{1}{4}} = \frac{2m}{3}(12a_1 + \sqrt{3}\,g)$ (A)

Then, for P (in the initial position):



Vert: $3mg - T_1 \cos\left(\frac{\pi}{6}\right) - T_2 \cos\left(\frac{\pi}{6}\right) = (3m)a_2$

Horiz: $T_1 \cos\left(\frac{\pi}{3}\right) - T_2 \cos\left(\frac{\pi}{3}\right) = (3m)a_1$, so that $T_1\left(\frac{\sqrt{3}}{2}\right) + T_2\left(\frac{\sqrt{3}}{2}\right) = 3m(g - a_2)$, or $T_1 + T_2 = \frac{6m}{\sqrt{3}}(g - a_2)$

and $T_1\left(\frac{1}{2}\right) - T_2\left(\frac{1}{2}\right) = 3ma_1$, or $T_1 - T_2 = 6ma_1$

Then, eliminating T_1: $2T_2 = \frac{6m}{\sqrt{3}}\left(g - a_2 - \sqrt{3}\,a_1\right)$,

so that $T_2 = m\sqrt{3}\left(g - a_2 - \sqrt{3}\,a_1\right)$ (B)

Equating (A) and (B) then gives:

$\frac{2m}{3}\left(12a_1 + \sqrt{3}\,g\right) = m\sqrt{3}\left(g - a_2 - \sqrt{3}\,a_1\right)$,

so that, as $a_1 = \sqrt{3}a_2$,

$24(\sqrt{3}a_2) + 2\sqrt{3}\,g = 3\sqrt{3}\left(g - a_2 - \sqrt{3}\,(\sqrt{3}a_2)\right)$

or $24a_2 + 2g = 3(g - 4a_2)$,

giving $36a_2 = g$, and so $a_2 = \frac{g}{36}$

And hence the magnitude of the initial acceleration is:

$\sqrt{{a_1}^2 + {a_2}^2} = \sqrt{3{a_2}^2 + {a_2}^2} = 2a_2 = \frac{g}{18}$, as required.