STEP - Misc. Topic Notes (6 pages; 2/6/23)

Contents

(A) Tests for divisibility

(1) If the sum of the digits of a number is a multiple of 3, then the number itself is a multiple of 3; and similarly for 9.

(2) $11 \times 325847 = 3584317$

and $3 - 5 + 8 - 4 + 3 - 1 + 7 = 11$, which is a multiple of 11

This is true in all cases: If $a - b + c - d + \cdots - z$ is a multiple of 11, then $abcd \dots z$ is a multiple of 11.

[and also for $a - b + c - d + \cdots + y$]

(B) Weak & Strong Induction

[$P(k)$ is the proposition that a particular result is true for $n = k$]

'Weak' induction is just the ordinary method

'Strong' induction is where we show that if $P(k-m)$,

$P(k-m+1), \ldots P(k)$ are correct, then $P(k+1)$ will be correct.
We then have to establish that $P(1), P(2), \ldots P(m+1)$ are correct.
(Weak induction corresponds to $m = 0$.)

Example: g_n is defined recursively as $(n^3 - 3n^2 + 2n)g_{n-3}$ for $n \geq 4$, and $g_1 = 1, g_2 = 2, g_3 = 6$

Show that $g_n = n!$ for $n \geq 1$

Solution

Assume that the result is true for $n = k-2, k-1$ & k.

Then $g_{k+1} = \left((k+1)^3 - 3(k+1)^2 + 2(k+1)\right)g_{k-2}$

$= (k+1)(k^2 + 2k + 1 - 3k - 3 + 2)(k-2)!$

$= (k+1)(k^2 - k)(k-2)!$

$= (k+1)k(k-1)(k-2)!$

$= (k+1)!$

So that the result is true for $n = k+1$ if it is true for

$n = k-2, k-1$ & k.

As it is true for $n = 1, 2$ & 3, it is therefore true for $n = 4, 5, \ldots$, and hence, by the principle of induction, it is true for all positive integers.

(C) Series

(1) $\sum_{r=1}^{n} r = 1 + 2 + 3 + \cdots + n = \frac{1}{2}\, n(n+1)$

[Informal proof: The average size of the terms being added is $\frac{1}{2}(1+n)$, and there are n terms.]

(2) See STEP 2008, P3, Q2 for a method to obtain $S_k(n) = \sum_{r=1}^{n} r^k$ for any n.

For example, $S_4(n) = \frac{1}{30} n(n+1)(2n+1)(3n^2+3n-1)$

(3) Taylor & Maclaurin expansions

(i) Maclaurin: $f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \cdots$

(ii) Taylor I: $f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots$

(iii) Taylor II: $f(x+a) = f(a) + xf'(a) + \frac{x^2}{2!} f''(a) + \cdots$

[$x = 0$ gives the Maclaurin expansion]

(D) Factorisations

(1)(i) $x^2 - y^2 = (x+y)(x-y)$

(ii) $x^3 - y^3 = (x-y)(x^2+xy+y^2)$

[Let $f(x) = x^3 - y^3$. Then $f(y) = 0$, and so $x - y$ is a factor of $x^3 - y^3$, by the Factor Theorem.]

$x^3 + y^3 = (x+y)(x^2-xy+y^2)$

(iii) $x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})$

or $(x+y)(x^{n-1}-x^{n-2}y+\cdots+xy^{n-2}-y^{n-1})$, if n is even

$x^n+y^n=(x+y)(x^{n-1}-x^{n-2}y+\cdots-xy^{n-2}+y^{n-1})$ if n is odd

(2) Let $f(n)$ be the number of factors of n (including 1).

If $n=pq$, where p & q have no common factors (other than 1), then $f(n)=f(p)f(q)$.

[eg $100=2^2\times 5^2$; factors are obtained from $\{1,2,4\}$ with $\{1,5,25\}$, giving a total of $3\times 3=9$ factors: $1,5,25,2,10,50,4,20,100$]

(E) Integer solutions

eg $xy-8x+6y=90$

can be rearranged to $(x+6)(y-8)=42$

(F) Trinomial expansions

(i) $(a+b+c)^2=(a^2+b^2+c^2)+2(ab+ac+bc)$

(ii) $(a+b+c)^3=(a^3+b^3+c^3)$

$+3(a^2b+a^2c+b^2a+b^2c+c^2a+c^2b)$

$+6abc$

(iii) $(a+b+c)^4=(a^4+b^4+c^4)$

$+4(a^3b+a^3c+b^3a+b^3c+c^3a+c^3b)$

$+6(a^2b^2+a^2c^2+b^2c^2)+12(a^2bc+b^2ac+c^2ab)$

(iv) $(a+b+c)^n=\sum_{\substack{i,j,k \\ (i+j+k=n)}}\binom{n}{i,j,k}a^ib^jc^k$, where $\binom{n}{i,j,k} = \frac{n!}{i!j!k!}$

(G) Equating coefficients

Example: To divide $f(x) = x^3 + x^2 - 11x + 10$ by $x - 2$

First of all, $f(2) = 8 + 4 - 22 + 10 = 0$, so that there is no remainder.

Then $x^3 + x^2 - 11x + 10 = (x - 2)(x^2 + ax - 5)$

Equating coefficients of x^2: $1 = a - 2$, so that $a = 3$

(Check: Equating coefficients of x: $-11 = -5 - 2a$, so that $a = 3$)

This method is usually quicker than long division.

(H) Polynomials

(1) Integer roots

Let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$

where $n \geq 2$ and the a_i are integers, with $a_0 \neq 0$.

Then it can be shown that any rational root of the equation $f(x) = 0$ will be an integer.

Proof

Suppose that there is a rational root $\frac{p}{q}$, where p & q are integers with no common factor greater than 1 and $q > 0$.

Then $\left(\frac{p}{q}\right)^n + a_{n-1}\left(\frac{p}{q}\right)^{n-1} + \cdots + a_2\left(\frac{p}{q}\right)^2 + a_1\left(\frac{p}{q}\right) + a_0 = 0$

and, multiplying by q^{n-1}:

$$\frac{p^n}{q} + a_{n-1}p^{n-1} + a_{n-2}p^{n-2}q + \cdots + a_1pq^{n-2} + a_1q^{n-1} = 0$$

Then, as all the terms from $a_{n-1}p^{n-1}$ onwards are integers, it follows that $\frac{p^n}{q}$ is also an integer, and hence $q = 1$ (as p & q have no common factor greater than 1), and the root is an integer.

(I) Hyperbolic Functions

$arsinhx = \ln\left(x + \sqrt{x^2 + 1}\right)$; $arcoshx = \ln\left(x + \sqrt{x^2 - 1}\right)$

Note that $coshy = x \Rightarrow y = \pm arcoshx = \pm\ln\left(x + \sqrt{x^2 - 1}\right)$,

which can be shown to equal $\ln\left(x \pm \sqrt{x^2 - 1}\right)$

[though note that $-\ln\,(a + b) \neq \ln\,(a - b)$ in general]