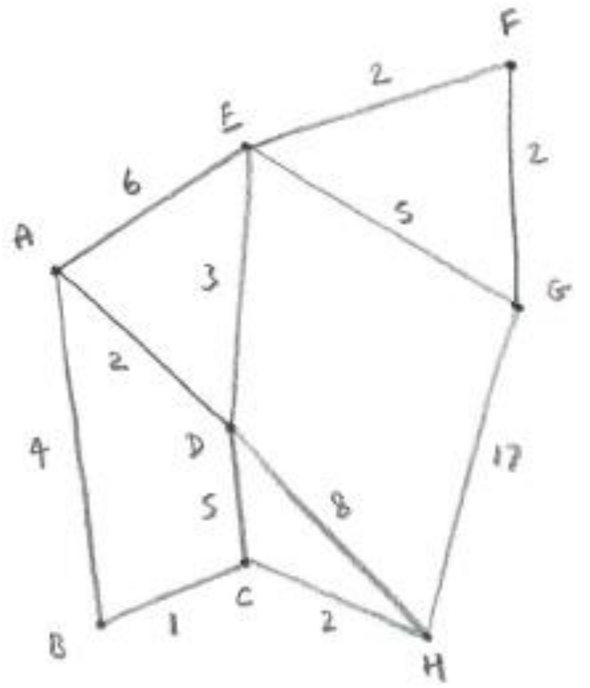
Route Inspection – Q2 (24/11/23)

Arnold, who is a railway enthusiast, wishes to travel along each stretch of railway linking the cities A-H of a particular country - as shown in the diagram, with the times (in hours) for each stretch. The total time for all the stretches is 57 hours.



(i) Initially he plans to set out from A and return to A. Find a route that covers each stretch of railway at least once, in the shortest possible time, and find the time taken.

(ii) There is a change of plan, and now Arnold wishes to start at A and finish at H (still covering each stretch at least once). Find the new time taken for the quickest route.

(iii) Arnold's wife wants the time reduced. If he has a free choice as to the starting and finishing cities, which should he choose, and what will the new time be?

Solution

(i) First of all, the orders of each node are established:

A 3 B 2 C 3 D 4 E 4 F 2 G 3 H 3

Thus there are 4 nodes of odd order: A, C, G and H.

In order to create an Eulerian graph (one where we can return to our starting position, having travelled along each arc exactly once), the nodes of odd order need to be converted to even order, by adding in extra arcs (which will be repeats of some of the existing arcs).

eg we might join up A & C, and then G & H

The other possibilities are: AG & CH, and AH & CG.

In joining up these nodes, we need to use paths between them which have the shortest total weight.

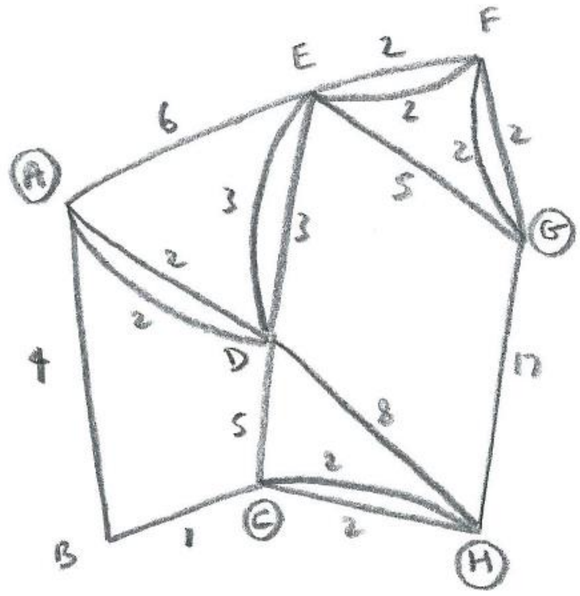
These total weights are:

AC & GH: 5[ABC]+14[GFEDCH] = 19

AG & CH: 9[ADEFG]+2[CH] = 11

AH & CG: 7[ABCH]+12[CDEFG] = 19

Thus the best option is AG & CH, which involving repeating the arcs AD, DE, EF, FG & CH (as shown below).



Notes

(a) There may be more than one shortest route, and it is not necessary to repeat the arcs AD, DE, EF and FG in this order.

(b) Notice that by adding in the extra arcs, D, E & F remain of even order.

One possibility is ABCDADEDHCHGEFGFEA

repeated arcs: AD ED CH GFE

The time taken is then $57 + 11 = 68$ hours

(ii) Starting at A and finishing at H means that nodes A and H can remain of odd order, so that only C & G need to be joined up.

This gives rise to an additional 12 hours (adding CDEFG), and hence the total time is now $57 + 12 = 69$ hours.

(iii) Of the available options for joining up nodes of odd order, CH has the shortest weight (of 2). Hence the best option is to start at A and finish at G, or vice versa, so that C and H have to be joined up.

This gives a total time of $57 + 2 = 59$ hours.