Rotating Bodies - Examples (6 pages; 16/3/18)

STEP 3, 2006, Q10

10 A disc rotates freely in a horizontal plane about a vertical axis through its centre. The moment of inertia of the disc about this axis is mk^2 (where $k > 0$). Along one diameter is a smooth narrow groove in which a particle of mass m slides freely. At time $t = 0$, the disc is rotating with angular speed Ω, and the particle is a distance a from the axis and is moving with speed V along the groove, towards the axis, where $k^2V^2 = \Omega^2a^2(k^2 + a^2)$.

Show that, at a later time t, while the particle is still moving towards the axis, the angular speed ω of the disc and the distance r of the particle from the axis are related by

$$\omega = \frac{\Omega(k^2 + a^2)}{k^2 + r^2} \quad \text{and} \quad \left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 = \frac{\Omega^2r^2(k^2 + a^2)^2}{k^2(k^2 + r^2)}.$$

Deduce that

$$k\frac{\mathrm{d}r}{\mathrm{d}\theta} = -r(k^2 + r^2)^{\frac{1}{2}},$$

where θ is the angle through which the disc has turned by time t.

By making the substitution $u = k/r$, or otherwise, show that $r\sinh(\theta + \alpha) = k$, where $\sinh\alpha = k/a$. Deduce that the particle never reaches the axis.

Solution

Conservation of energy:

$$\frac{1}{2}mV^2 + \frac{1}{2}mk^2\Omega^2 + \frac{1}{2}ma^2\Omega^2 = \frac{1}{2}m\left(\frac{dr}{dt}\right)^2 + \frac{1}{2}mk^2\omega^2 + \frac{1}{2}mr^2\omega^2$$

(ma^2 & mr^2 being the moment of inertia of the particle initially and at time t)

$$\Rightarrow V^2 + k^2\Omega^2 + a^2\Omega^2 = \left(\frac{dr}{dt}\right)^2 + k^2\omega^2 + r^2\omega^2$$

$$\Rightarrow \left(\frac{dr}{dt}\right)^2 = V^2 + \Omega^2(k^2 + a^2) - \omega^2(k^2 + r^2)$$

[noting the forms of the expressions to be established]

Also, from conservation of angular momentum:

$(mk^2 + ma^2)\Omega = (mk^2 + mr^2)\omega$

$\Rightarrow \omega = \frac{\Omega(k^2+a^2)}{k^2+r^2}$, as required (1)

Then $\left(\frac{dr}{dt}\right)^2 = \frac{\Omega^2 a^2(k^2+a^2)}{k^2} + \Omega^2(k^2 + a^2) - \frac{\Omega^2(k^2+a^2)^2}{k^2+r^2}$

$= \frac{\Omega^2(k^2+a^2)}{k^2(k^2+r^2)}\{a^2(k^2 + r^2) + k^2(k^2 + r^2) - k^2(k^2 + a^2)\}$

$= \frac{\Omega^2(k^2+a^2)}{k^2(k^2+r^2)}\{r^2(a^2 + k^2)\} = \frac{\Omega^2 r^2(k^2+a^2)^2}{k^2(k^2+r^2)}$, as required (2)

$\frac{dr}{d\theta} = \frac{dr}{dt}.\frac{dt}{d\theta} = \frac{dr}{dt} .\frac{1}{\omega}$

Then from (1) & (2), $\left(\frac{dr}{d\theta}\right)^2 = \frac{1}{\omega^2}\left(\frac{dr}{dt}\right)^2 = \left(\frac{k^2+r^2}{\Omega(k^2+a^2)}\right)^2 \frac{\Omega^2 r^2(k^2+a^2)^2}{k^2(k^2+r^2)}$

$= \frac{r^2(k^2+r^2)}{k^2}$

As the particle is moving towards 0, $\frac{dr}{dt} < 0$, and hence

$\frac{dr}{d\theta} = \frac{dr}{dt} .\frac{1}{\omega} < 0$

So $\frac{dr}{d\theta} = -\frac{r(k^2+r^2)^{1/2}}{k}$, giving $k\frac{dr}{d\theta} = -r(k^2 + r^2)^{\frac{1}{2}}$, as required.

$\frac{d\theta}{dr} = -\frac{k}{r\sqrt{k^2+r^2}}$ and hence $\theta - 0 = -k\int_a^r \frac{1}{R\sqrt{k^2+R^2}}dR$

[as an alternative to establishing the constant of integration explicitly; R is a dummy variable]

Let $u = k/R$, so that $du = -\frac{k}{R^2}dR$

and $\theta = \int_{k/a}^{k/r} \frac{u}{\sqrt{k^2+R^2}}\left(\frac{R^2}{k}\right) du = \int_{k/a}^{k/r} \frac{u}{\sqrt{u^2+1}}\left(\frac{R}{k}\right) du$

$= \int_{\frac{k}{a}}^{\frac{k}{r}} \frac{1}{\sqrt{u^2+1}} du = [arsinhu]_{k/a}^{k/r}$

So $\theta = arsinh\left(\frac{k}{r}\right) - arsinh(\frac{k}{a})$

and $\theta + \alpha = arsinh\left(\frac{k}{r}\right)$, where $sinh\alpha = k/a$

Thus $rsinh(\theta + \alpha) = k$, as required.

The particle reaches the axis when $r = 0$, which requires $\theta = \infty$; ie it never happens.

STEP 3, 2012, Q9

9 A pulley consists of a disc of radius r with centre O and a light thin axle through O perpendicular to the plane of the disc. The disc is non-uniform, its mass is M and its centre of mass is at O. The axle is fixed and horizontal.

Two particles, of masses m_1 and m_2 where $m_1 > m_2$, are connected by a light inextensible string which passes over the pulley. The contact between the string and the pulley is rough enough to prevent the string sliding. The pulley turns and the vertical force on the axle is found, by measurement, to be $P + Mg$.

(i) The moment of inertia of the pulley about its axle is calculated assuming that the pulley rotates without friction about its axle. Show that the calculated value is

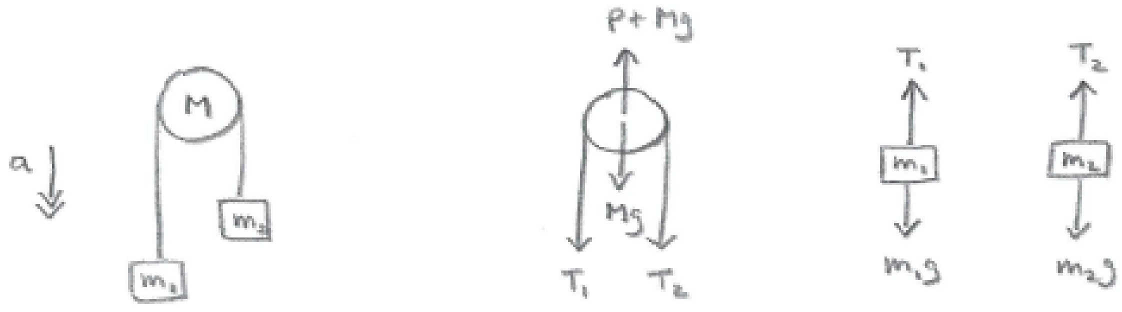
$$\frac{((m_1 + m_2)P - 4m_1m_2g)r^2}{(m_1 + m_2)g - P}. \qquad (*)$$

(ii) Instead, the moment of inertia of the pulley about its axle is calculated assuming that a couple of magnitude C due to friction acts on the axle of the pulley. Determine whether this calculated value is greater or smaller than $(*)$.

Show that $C < (m_1 - m_2)rg$.

Solution

(i)



N2L for the two particles gives:

$m_1 g - T_1 = m_1 a$ (1) & $T_2 - m_2 g = m_2 a$ (2)

(with obvious notation [which unfortunately would have to be defined in the exam])

[As the contact between the string and the pulley is rough, there is a frictional force (F) on the string. Considering the forces on the string, $T_1 - T_2 - F = ma$, where m, the mass of the string, is negligible, so that $T_1 = T_2 + F$; ie $T_1 \neq T_2$. In A Level questions, without this friction, the assumption is made that $T_1 = T_2$.]

For the pulley, total moments of external forces about O $= I\ddot{\theta}$,

so that $T_1 r - T_2 r = I\left(\frac{a}{r}\right) \Rightarrow T_1 - T_2 = I\left(\frac{a}{r^2}\right)$ (3)

[$\dot{\theta} = \frac{v}{r}$, so that $\ddot{\theta} = \frac{a}{r}$]

Also, resolving vertically for the pulley:

$P + Mg = T_1 + T_2 + Mg \Rightarrow P = T_1 + T_2$ (4)

(3) $\Rightarrow I = \frac{r^2}{a}(T_1 - T_2)$ (5)

(1) $\Rightarrow T_1 = m_1(g - a)$ (6)

$(2) \Rightarrow T_2 = m_2(g + a)$ (7)

$(4),(6),(7) \Rightarrow P = (m_1 + m_2)g + a(m_2 - m_1)$

$\Rightarrow a = \frac{P-(m_1+m_2)g}{m_2-m_1}$ (8)

$(5),(6),(7) \Rightarrow I = \frac{r^2}{a}\left(g(m_1 - m_2) - a(m_1 + m_2)\right)$

Then, from (8):

$I = -\frac{r^2 g(m_1-m_2)^2}{P-(m_1+m_2)g} - r^2(m_1 + m_2)$

$= \frac{r^2\{g(m_1-m_2)^2-(m_1+m_2)[(m_1+m_2)g-P]\}}{(m_1+m_2)g-P}$

$= \frac{r^2\{(m_1+m_2)P+g[(m_1-m_2)^2-(m_1+m_2)^2]\}}{(m_1+m_2)g-P}$

$= \frac{r^2\{(m_1+m_2)P-4m_1m_2g\}}{(m_1+m_2)g-P}$, as required

(ii) Let I_0 & I_1 be the old and new moments of inertia.

The only change is that equation (3) becomes

$(T_1 - T_2)r - C = I_1\left(\frac{a}{r}\right)$, as friction opposes motion (9)

Then, since $(T_1 - T_2)r = I_0\left(\frac{a}{r}\right)$,

$I_1 = \left\{I_0\left(\frac{a}{r}\right) - C\right\}\left(\frac{r}{a}\right) = I_0 - \frac{Cr}{a}$

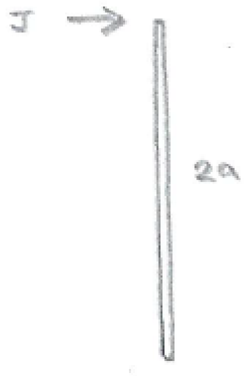
ie the new moment of inertia is smaller than the old value.

From (9), as $I_1 > 0, C < (T_1 - T_2)r = m_1(g - a)r - m_2(g + a)r$

$= (m_1 - m_2)rg - ar(m_1 - m_2) < (m_1 - m_2)rg$, as required

(as $m_1 > m_2$ & $a > 0$)

Example: An impulse J is applied to one end of a thin, uniform rod of length $2a$ and mass m, as shown below. Describe the resulting motion.



Solution

By conservation of linear momentum, if v is the velocity of the centre of mass of the rod after the impulse, then:

$J = mv \quad (1)$

And by conservation of angular momentum, if ω is the angular velocity about the centre of mass after the impulse, then

$aJ = I\omega \quad (2)$,

where I, the moment of inertia of the rod about an axis through the centre of mass, perpendicular to the rod $= \frac{1}{3}ma^2$

So, the motion of the rod after the impulse is a combination of a velocity of $v = \frac{J}{m}$ in the direction of the impulse, together with a rotation about the centre of mass, with angular velocity

$\omega = \frac{aJ}{\left(\frac{1}{3}ma^2\right)} = \frac{3J}{ma}$