Rolling Wheel – Speed of point on circumference

(5 pages; 1/2/21)

(A) Speed relative to the ground - Part 1

Assume that a light hoop (ie of negligible mass) is rolling (without slipping) at a constant speed v on horizontal ground. How fast is the point P on the hoop (in Figure 1) moving relative to the ground (a) when it is touching the ground, and (b) when it is at the top?

Suppose that the hoop is rotating in a clockwise sense and has radius a. Let θ be as in the diagram.

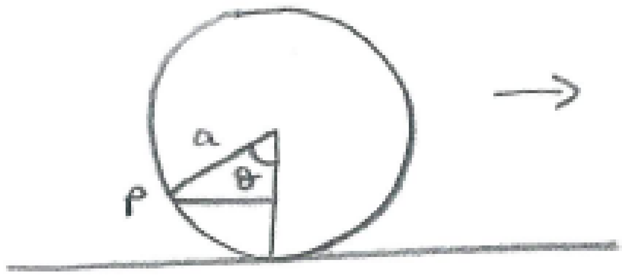

Figure 1

As P leaves the ground, it moves round to the left, and has speed a$\dot{\theta}$ along the circumference of the hoop, relative to the centre of the hoop. [The arc length travelled as the angle increases by θ rad is $a\theta$, and the tangential speed is $\frac{d}{dt}(a\theta) = a\dot{\theta}$ (where $\dot{\theta} \equiv \frac{d\theta}{dt}$)]
The ground that the hoop covers is equal to the distance that P moves along the circumference, and hence the centre of the hoop is also moving with speed a$\dot{\theta}$ (but to the right, and relative to the ground), so that $v = a\dot{\theta}$.

The motion of P relative to the ground has two components: its motion relative to the centre of the hoop, and the motion of the centre of the hoop relative to the ground. When P is touching the ground, its speed relative to the ground is therefore

$-a\dot{\theta} + a\dot{\theta} = 0$ (if motion to the right is considered to be positive); ie P is stationary!

At the top, P has speed $a\dot{\theta} + a\dot{\theta} = 2a\dot{\theta} = v$, relative to the ground.

The fact that the hoop is rolling rather than slipping means that it is effectively toppling continually about the point of contact with the ground. (Consider a circular star-like structure, with a large number of limbs.)

(B) Speed relative to the ground - Part 2

For the same hoop, how fast is the general point P (in Figure 2) moving relative to the ground – in terms of θ? And (separately) in terms of h (the height above the ground)?

Let this speed be v_P (in the direction indicated in Figure 2).

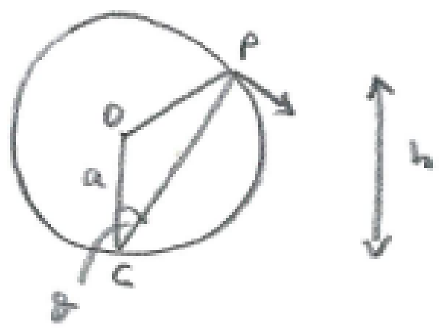

Figure 2

We can use the fact that the point of contact of the hoop with the ground is the 'instantaneous centre of rotation' of the hoop: every point on the hoop is (instantaneously) rotating about C with the same angular speed, ω say.

Every point is therefore instantaneously moving at right angles to the line joining it to C; PC in the case of the point P.

The point at the top of the hoop has speed $2v$ relative to the ground (from Part 1), so that $\omega = \frac{2v}{2a} = \frac{v}{a}$ (1)

[since, in general, the tangential speed v and the angular speed ω are related by $v = \omega r$, where r is the radius of the circular path being followed]

For P, $\omega = \frac{v_P}{PC}$ (2)

As the triangle OPC is isosceles , $PC = 2acos\theta$

Hence, from (1) & (2), $v_P = \frac{v}{a}PC = 2vcos\theta$

Also, $h = PCcos\theta$ (drawing a line from P to the ground, parallel to OC) $= 2acos^2\theta$, so that $v_P = 2v\sqrt{\frac{h}{2a}} = v\sqrt{\frac{2h}{a}}$

Note that both results agree with Part 1 when P is at the top or bottom of the hoop.

Note also that, in Part 1, the angular speed of P about the centre O was $\omega = \frac{v}{a} = \dot{\theta}$; ie the angular speed about the instantaneous centre of rotation is the same as the angular speed about the centre.

(C) Speed relative to the ground - Part 3

Alternative method:

The velocity of point P has two components - translational and rotational.

The translational component is horizontal and equal to v (the same as the centre of the hoop).

The rotational component is perpendicular to OP and is equal to

$a\dot{\theta} = v$ (also).

This immediately gives the speed relative to the ground when P is at the top or bottom.

For general P (referring to Figure 3):

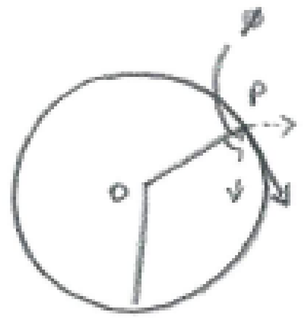

Figure 3

The rotational component can itself be resolved into horizontal and vertical components, as follows:

$\phi = \pi - \angle COP = \pi - (\pi - 2\theta) = 2\theta$

(referring to the isosceles triangle (and the point C) in Figure 2)

Then its vertical component $= vcos\left(\frac{\pi}{2} - \phi\right) = vsin\phi = vsin(2\theta)$

and its horizontal component $= vcos(2\theta)$

Adding in the translational component, gives:

total horizontal component $= v(1 + cos2\theta)$

and total vertical component $= vsin2\theta$,

so that the magnitude of the velocity of P is:

$v\sqrt{(1 + cos2\theta)^2 + sin^2 2\theta}$

$= v\sqrt{(1 + cos^2\theta - \ sin^2\theta)^2 + 4sin^2\theta cos^2\theta}$

$= v\sqrt{(2cos^2\theta)^2 + 4sin^2\theta cos^2\theta}$

$= v\sqrt{4cos^4\theta + 4sin^2\theta cos^2\theta}$

$= v\sqrt{4cos^2\theta(cos^2\theta + \ sin^2\theta)}$

$= 2vcos\theta,$ as before