Proof (4 pages; 16/1/23)

(1) If and only if

Example: A quadratic equation has no real roots (A) if and only if its discriminant is negative (B).

We can then write the following (true) statements:

$A \Leftrightarrow B$ (A is true **if and only if** B is true)

$A \Leftarrow B$ (A is true **if** B is true, or B is a sufficient condition for A)

$A \Rightarrow B$ (A is true **only if** B is true, or B is a necessary condition for A)

(2) $A \Leftrightarrow B$ can be read in the following ways:

(a) "A implies, and is implied by, B"

(b) "A is true if and only if B is true" (abbreviated to "A iff B")

(c) "A is a necessary and sufficient condition for B"

(or alternatively "B is a necessary and sufficient condition for A")

But note that the "implies" part of (a) corresponds to the "only if" part of (b), and to the "sufficient" part of (c); ie (b) and (c) are the wrong way round, compared with (a).

Notes

(i) $A \Rightarrow B$ is sometimes read as "If A then B"

(ii) $A \Leftarrow B$ (or $B \Rightarrow A$) is sometimes read as "A if B"

(iii) $A \Rightarrow B$ (or $B \Leftarrow A$) is sometimes read as A only if B

(3) Methods of proof of $A \Leftrightarrow B$

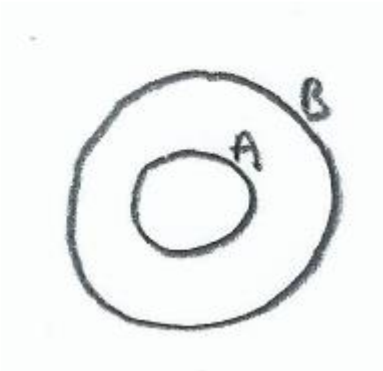
In some situations, it is possible to produce a chain of clearly equivalent statements, such as: $A \Leftrightarrow X \Leftrightarrow Y \Leftrightarrow B$. In others it may be necessary to prove separately that $A \Rightarrow B$ and $B \Rightarrow A$.

As an alternative to proving that $B \Rightarrow A$, we can instead prove that $A' \Rightarrow B'$. To show that $A' \Rightarrow B'$ is equivalent to $B \Rightarrow A$:

If B is true, suppose that A is not true. Then, as $A' \Rightarrow B'$, there is a contradiction, as B is true. So A must be true, and hence $B \Rightarrow A$.

[$A' \Rightarrow B'$ is known as a 'proof by contrapositive']

(4) Venn diagram interpretation



$A \Leftrightarrow B$ means that the gap between A and B is an empty set (ie there are no events represented by this gap)

$A \Rightarrow B$ means A is contained in B

$A' \Rightarrow B'$ means if an event is outside of A, then it has to be outside of B

Together, these mean that the gap between A and B is an empty set.

Also, consider the following non-mathematical example: suppose that A is "Lives in London", and B is "Lives in England".

Then $A \Rightarrow B$, but $B \nRightarrow A$. Also $A' \nRightarrow B'$

(5) Examples

(a) Let C be the event that two triangles are congruent, and let S be the event that they are similar. Then $C \Rightarrow S$, but $S \nRightarrow C$.

(b) If n is a positive integer, and n^2 is odd (A), prove that n is odd (B). [Result to prove: $A \Rightarrow B$]

Solution

Method 1: Proof by contradiction

Suppose that n is even. Then $n = 2m$, for some positive integer m.

But then $n^2 = (2m)^2 = 4m^2$, which is divisible by 2, and hence even. This contradicts the fact that n^2 is odd, and so n must be odd.

Method 2: Using contrapositive

To prove that $B' \Rightarrow A'$:

Suppose that n is even. Then (as before) n^2 is even, so that A' holds.

(c) Let A be the statement: The transformation represented by the 2×2 matrix $\underline{A}$ has a line of invariant points that does not pass through the Origin;

let B be the statement: $\underline{A} = \underline{I}$ (for 2×2 matrices)

It can be shown that $A \Leftrightarrow B$

[$B \Rightarrow A$ follows from the fact that every point is invariant for the identity transformation; for a proof that $A \Rightarrow B$, see sol'n to STEP 2019, P3, Q3(i)]

(6) Consider the following (unsatisfactory) proof:

"To show that $tan\theta + cot\theta \equiv sec\theta cosec\theta$ [A]:

$tan\theta + cot\theta \equiv sec\theta cosec\theta \Rightarrow tan\theta + cot\theta - sec\theta cosec\theta \equiv 0$

$\Rightarrow \frac{sin^2\theta + cos^2\theta - 1}{cos\theta sin\theta} = 0$

$\Rightarrow 0 = 0$ [B] ($cos\theta sin\theta \neq 0$, as $sec\theta$ & $cosec\theta$ are assumed to be defined, so that $cos\theta$ & $sin\theta$ are both non-zero)"

This only shows that [A] ⇒ [B], whereas we want to show that [B] ⇒ [A]. The proof can be salvaged by replacing ⇒ by ⇔ (as equivalence is clearly true at each stage), though the use of

"$0 = 0$" isn't usually thought to be that elegant. We would still need to make it clear that we had shown that [B] ⇒ [A].

(7) Converse, Contrapositive and Inverse

(i) The converse of $X \Rightarrow Y$ is $Y \Rightarrow$ X (or $X \Leftarrow Y$)

(ii) The contrapositive of $X \Rightarrow Y$ is $Y' \Rightarrow$ X′. This is mathematically equivalent to $X \Rightarrow Y$.

(iii) The inverse of $X \Rightarrow Y$ is $X' \Rightarrow Y'$. This is mathematically equivalent to $Y \Rightarrow$ X (ie the converse of $X \Rightarrow Y$).