Probability & Counting Approaches (12 pages; 21/3/24)

See also: Prob. & Stats – "Selections"

Contents

(1) Overview of Approaches and Devices

(A) Symmetry / lateral thinking

[Although an argument along these lines may save a lot of time, it has to be 'convincing' for exam purposes, and so may be risky to some extent.]

Refer to the following example(s):

Example 1, Approach 1

Example 7, Approach 1

Example 8

(B) 'One step at a time'

Refer to the following example(s):

Example 3, Approach 1

Example 3, Approach 1a

(C) 'Basic definition of probability'

$$\frac{\text{number of favourable outcomes}}{\text{number of possible outcomes}}$$

(provided the outcomes are equally likely)

Refer to the following example(s):

Example 2, Approach 1

Example 3, Approach 2

Example 5, Approach 2

Example 6, Approach 2

(D) 'Case by case'

Refer to the following example(s):

Example 2, Approach 1

Example 6, Approach 1

(E) Conditional Probability

Refer to the following example(s):

Example 1, Approach 2

Example 2, Approach 2

Example 7, Approach 2

Example 8

(F) Counting devices

(F.1) Solve problem for specific order, and then consider number of possible orders.

Refer to the following example(s):

Example 3, Approach 1a

Example 5, Approach 1

(F.2) 'One step at a time' (see (B))

(F.3) 'Case by case'

(F.4) Initially include non-permissible cases, and then deduct them.

(F.5) If items have to be next to each other, combine them into a single block of r items, and multiply by r!

(F.6) For r indistinguishable items, assume initially that they are different, and then remove duplication by dividing by r!

(F.7) Find a systematic way of listing the possible situations

(G) Recurrence relation

Refer to the following example(s):

Example 4

See also: STEP 2021, P2, Q11

(H) Venn diagram notation

(2) Examples

Example 1

A fair die is thrown repeatedly.

To find P(At least one 5 arises before the 1st 6)

Approach 1: Symmetry

This event is the same as "a 5 occurs before a 6", so the probability is $\frac{1}{2}$.

Approach 2: Conditional probability

$P(1st\ 6\ arises\ on\ the\ rth\ throw) = \left(\frac{5}{6}\right)^{r-1}.\frac{1}{6}$

So P(At least one 5 arises before the 1st 6)

$= \sum_{r=2}^{\infty}\{P(1st\ 6\ arises\ on\ rth\ throw)[1 - P(no\ 5s\ arise\ in\ 1st$

$r - 1\ throws|no\ 6s\ arise\ in\ the\ 1st\ r - 1\ throws)]\}$

[Noting that, if we know that a 6 has not arisen in the 1st $r - 1$ throws, then at each throw there are only 5 possible (and equally likely) outcomes.]

$= \sum_{r=2}^{\infty}\left(\frac{5}{6}\right)^{r-1}.\frac{1}{6}(1 - \left(\frac{4}{5}\right)^{r-1})$

$= \frac{5}{6}.\frac{1}{6}.\frac{1}{1-\frac{5}{6}} - \frac{1}{6}\sum_{r=2}^{\infty}\left(\frac{4}{6}\right)^{r-1}$

$= \frac{5}{6} - \frac{1}{6}.\frac{4}{6}.\frac{1}{1-\frac{4}{6}}$

$= \frac{5}{6} - \frac{4}{12} = \frac{1}{2}$

Example 2

A fair die is thrown repeatedly.

To find P(Exactly one 5 arises before the 1st 6)

Approach 1: 'Case by case' (using 'Basic Definition')

Considering separately the cases where the 1st 6 arises on the 2nd, 3rd, ... throws (as these are mutually exclusive and exhaustive events; ie they don't overlap and they include all possibilities)

Now $P(one\ 5\ and\ no\ sixes\ arise\ \ in\ the\ 1st\ r-1\ throws)$

$$= \frac{no.\ of\ ways\ of\ one\ 5\ and\ no\ sixes\ arising\ \ in\ the\ 1st\ r-1\ throws}{no.of\ possibilities\ for\ 1st\ r-1\ throws}$$

$$= \frac{(r-1)\times 4^{r-2}}{6^{r-1}}$$

[Consider eg 5412 ... 3: there are 4^{r-2} ways of filling the last $r-2$ places with the numbers 1-4; and there are $r-1$ possible positions for the 5]

So P(Exactly one 5 arises before the 1st 6)

$= \sum_{r=2}^{\infty}\{P(one\ 5\ and\ no\ sixes\ arise\ \ in\ the\ 1st\ r-1\ throws).$

$P(a\ 6\ arises\ on\ the\ rth\ throw)\}$

$$= \sum_{r=2}^{\infty}\frac{(r-1)\times 4^{r-2}}{6^{r-1}}\ .\frac{1}{6} = \frac{1}{36}\sum_{r=2}^{\infty}(r-1)\left(\frac{4}{6}\right)^{r-2}$$

$$= \frac{1}{36}\sum_{R=1}^{\infty} R\left(\frac{2}{3}\right)^{R-1}$$, where $R = r-1$ (*)

From Standard Results (1), $\sum_{r=1}^{\infty} ra^r = \frac{a}{(1-a)^2}$ (when $|a| < 1$),

so that $(*) = \frac{1}{36}\ .\frac{3}{2}\ \frac{\left(\frac{2}{3}\right)}{(1-\frac{2}{3})^2} = \frac{1}{36(\frac{1}{9})} = \frac{1}{4}$

Approach 2: Conditional Probability

P(Exactly one 5 arises before the 1st 6)

$$= \sum_{r=2}^{\infty} \{P(1st\ 6\ arises\ on\ rth\ throw)P(exactly\ one\ 5\ arises\ in\ the\ 1st$$

$$r-1\ throws|no\ 6s\ arise\ in\ the\ 1st\ r-1\ throws)\}$$

$$= \sum_{r=2}^{\infty} \left(\frac{5}{6}\right)^{r-1} . \frac{1}{6} . \binom{r-1}{1} \left(\frac{1}{5}\right)\left(\frac{4}{5}\right)^{r-2}$$

$$= \frac{5}{6} . \frac{1}{6} . \frac{1}{5} \sum_{r=2}^{\infty} (r-1) \left(\frac{4}{6}\right)^{r-2}$$

$$= \frac{1}{36} \sum_{R=1}^{\infty} R \left(\frac{2}{3}\right)^{R-1}$$ (where $R = r-1$) , as in Approach 1

Example 3

3 letters are selected from a bag containing the letters AAABBBCCC

To find P(3 different letters are chosen)

Approach 1: 'One step at a time'

P(3 different letters are chosen)

= P(any tablet is chosen initially)

× P(a different tablet is then chosen)

× P(a tablet different from the 1st 2 is then chosen)

$$= 1 \times \frac{6}{8} \times \frac{3}{7} = \frac{9}{28}$$

Approach 1a: 'One step at a time'

P(ABC are chosen – in that order) $= \frac{3}{9} \times \frac{3}{8} \times \frac{3}{7} = \frac{3}{7 \times 8}$

As there are 3! ways of ordering ABC,

P(3 different letters are chosen) $= \frac{3}{7\times8} \times 3! = \frac{9}{28}$

Approach 2: 'Basic definition'

P(3 different letters are chosen)

$$= \frac{\text{no. of ways of obtaining the letters ABC (where order doesn't matter)}}{\text{no. of ways of chosing 3 letters out of 9 (where order doesn't matter)}}$$

$$= \frac{\binom{3}{1}\times\binom{3}{1}\times\binom{3}{1}}{\binom{9}{3}} = \frac{27}{\left(\frac{9(8)(7)}{3!}\right)} = \frac{9}{28}$$

Example 4

A and B take it in turns to shoot arrows at a target, with A starting first. The probability that A hits the target is a and the probability that B hits the target is b. The winner is the person who hits the target first. Find the probability that A wins.

Solution

Let α be the probability that A wins.

Then $\alpha = \text{P(A wins on 1st attempt)} + \text{P(wins after 1st attempt)}$

$= a + (1-a)(1-b)\alpha$, so that $\alpha(1-(1-a)(1-b)) = a$, and
$\alpha = \frac{a}{1-(1-a)(1-b)}$

Example 5

n boys and 3 girls are to be seated in a row at random; K is the maximum consecutive number of girls in the row; find $P(K = 3)$

Approach 1

$P(GGGB \ldots B) = \frac{3}{n+3} \cdot \frac{2}{n+2} \cdot \frac{1}{n+1}$

There are $n + 1$ possibilities in total (the 1st G can be in positions 1 to $n + 1$), and they are all equally likely.

So $P(K = 3) = \frac{3}{n+3} \cdot \frac{2}{n+2} \cdot \frac{1}{n+1} \cdot (n + 1) = \frac{6}{(n+2)(n+3)}$

Approach 2 ('Basic definition')

There are $\binom{n+3}{3}$ equally likely ways of choosing the 3 positions for the Gs, and the Gs will be together in $n + 1$ of these (as in Approach 1).

So $P(K = 3) = \frac{n+1}{\binom{n+3}{3}} = \frac{(n+1)(3!)}{(n+3)(n+2)(n+1)} = \frac{6}{(n+2)(n+3)}$

Example 6

n boys and 3 girls are to be seated in a row at random; K is the maximum consecutive number of girls in the row; find $P(K = 1)$

Approach 1

Let $P = GB$

Case 1: The last child is not a girl.

Examples: $BBPBBPBP$, $BBPPP$B

Case 2: The last child is a girl.

Examples: $BBPBBPBG$, $BBPBBPG$

The number of possibilities for Case 1 (with n boys & 3 girls, and therefore 3 Ps & $(n-3)$ Bs) is $\binom{n}{3}$

The number of possibilities for Case 2

(with 2 Ps , $(n-2)$Bs & the G at the end) is $\binom{n}{2}$

All the possibilities have probability $\frac{3}{n+3} \cdot \frac{2}{n+2} \cdot \frac{1}{n+1}$ (from Example 5).

So $P(K=1) = \frac{6}{(n+3)(n+2)(n+1)}\left\{\binom{n}{3} + \binom{n}{2}\right\}$

$= \frac{6}{(n+3)(n+2)(n+1)}\left\{\frac{n(n-1)(n-2)}{3!} + \frac{n(n-1)}{2!}\right\}$

$= \frac{n(n-1)(n-2+3)}{(n+3)(n+2)(n+1)} = \frac{n(n-1))}{(n+3)(n+2)}$

Approach 2

Consider $XBXBX \dots BX$ (with alternating X & B).

3 of the $n+1$ Xs will be filled by the girls, with the remaining Xs being empty. This can be done in $\binom{n+1}{3}$ ways.

Overall there are $\binom{n+3}{3}$ ways of arranging the n boys and 3 girls.

So $P(K=1) = \frac{\binom{n+1}{3}}{\binom{n+3}{3}} = \frac{\left(\frac{(n+1)!}{3!(n-2)!}\right)}{\left(\frac{(n+3)!}{3!n!}\right)} = \frac{n(n-1)}{(n+3)(n+2)}$

Example 7

A bag contains N balls (where $N \geq 2$), of which n are white. Two balls are drawn from the bag without replacement. Show that the probability that the 1st ball is white is equal to the probability that the 2nd ball is white.

Approach 1: Symmetry

Drawing one ball and then another is no different from putting both hands into the bag and drawing a ball with each hand, but designating the right-hand ball as the 1st drawn. But alternatively we could have designated the left-hand ball as the 2nd drawn.

Approach 2: Conditional Probability

$P(1st\ is\ W) = \frac{n}{N}$

$P(2nd\ is\ W) = P(1st\ is\ W)P(2nd\ is\ W|1st\ is\ W)$

$+P(1st\ is\ not\ W)P(2nd\ is\ W|1st\ is\ not\ W)$

$= \left(\frac{n}{N}\right)\left(\frac{n-1}{N-1}\right) + \left(\frac{N-n}{N}\right)\left(\frac{n}{N-1}\right)$

$= \frac{n}{N(N-1)}(n - 1 + N - n) = \frac{n}{N} = P(1st\ is\ W)$, as required

Example 8

A (possibly biased) coin is tossed repeatedly in a game between A and B, with p being the probability of obtaining a head, and $q = 1 - p$ the probability of obtaining a tail. A wins if two successive heads appear, and B wins if two successive tails appear. It can be assumed that the game will end eventually. Find the probability that A wins.

Solution

Step 1

P(A wins |1st toss is H)

$= \text{P(2nd toss is H)}$

$+ \sum_{r=1}^{\infty} \text{P(2nd toss is T } and \text{ A wins on (2r + 2)nd toss}$

|1st toss is H)

$= p + \sum_{r=1}^{\infty} q(pq)^{r-1}p^2$

$= p + qp^2 . \frac{1}{1-pq}$

$= \frac{p(1-pq)+qp^2}{1-pq}$

$= \frac{p}{1-pq}$

Step 2

By symmetry, $\text{P(B wins |1st toss is T)} = \frac{q}{1-qp}$

And $\text{P(A wins |1st toss is T)} = 1 - \text{P(B wins |1st toss is T)}$

$= 1 - \frac{q}{1-qp} = \frac{1-qp-q}{1-qp} = \frac{p-qp}{1-qp} = \frac{p(1-q)}{1-qp} = \frac{p^2}{1-qp}$

[Alternatively, P(A wins |1st toss is T)

$= \text{P(2nd toss is H). P(A wins |1st toss is H)}$

$= p . \frac{p}{1-pq} = \frac{p^2}{1-qp}$]

Step 3

$P(A\ wins) = p.\text{P(A wins |1st toss is H)}$

$+q.\text{P(A wins |1st toss is T)}$

$= p.\frac{p}{1-pq} + q.\frac{p^2}{1-qp}$

$= \frac{p^2(1+q)}{1-pq}$

[Check: By symmetry, $P(B\ wins) = \frac{q^2(1+p)}{1-qp}$

and $\frac{p^2(1+q)}{1-pq} + \frac{q^2(1+p)}{1-qp} = \frac{p^2+p^2q+q^2+q^2p}{1-pq} = \frac{p^2+q^2+pq(p+q)}{1-pq}$

$= \frac{p^2+q^2+pq}{1-pq} = \frac{p(p+q)+q^2}{1-pq} = \frac{p+q^2}{1-pq} = \frac{1-q+q^2}{1-pq} = \frac{1-q(1-q)}{1-pq} = \frac{1-qp}{1-pq} = 1$]

Appendix: Standard results

(1) $\sum_{r=1}^{\infty} ra^r = a\frac{d}{da}\sum_{r=1}^{\infty} a^r = a\frac{d}{da}(\frac{a}{1-a})$ (when $|a| < 1$)

$= a\,.\frac{(1-a)-a(-1)}{(1-a)^2} = \frac{a}{(1-a)^2}$