Polynomials – Q7 (26/6/23)

If α, β and γ are the roots of the equation

$x^3 - 2x^2 - 4x + 5 = 0$,

find the equation with roots $\alpha + \beta\gamma$, $\beta + \alpha\gamma$ and $\gamma + \alpha\beta$.

Solution

Let the new equation be $x^3 + bx^2 + cx + d = 0$

Then $b = -(\alpha + \beta\gamma + \beta + \alpha\gamma + \gamma + \alpha\beta)$

$= -\sum \alpha - \sum \alpha\beta = -2 - (-4) = 2$

$c = (\alpha + \beta\gamma)(\beta + \alpha\gamma) + (\alpha + \beta\gamma)(\gamma + \alpha\beta) + (\beta + \alpha\gamma)(\gamma + \alpha\beta)$

$= (\alpha\beta + \alpha^2\gamma + \beta^2\gamma + \alpha\beta\gamma^2) + \ldots$

[By symmetry, this contains all the types of terms appearing in the full expansion, and there are $3(4) = 12$ terms.]

$= \sum \alpha\beta + \sum \alpha^2\beta + \sum \alpha\beta\gamma^2$

[As a check, this contains $3 + 6 + 3 = 12$ terms]

Thus $c = (-4) + \sum \alpha^2\beta + \alpha\beta\gamma \sum \alpha$

$(-4) + \sum \alpha^2\beta + (-5)(2) = -14 + \sum \alpha^2\beta$ (A)

[$\sum \alpha^2\beta$ to be found shortly]

And $d = -(\alpha + \beta\gamma)(\beta + \alpha\gamma)(\gamma + \alpha\beta)$

[this will give $2^3 = 8$ terms]

$= -(\alpha\beta\gamma + (\sum \alpha^2\beta^2) + \alpha^2\beta^2\gamma^2 + \sum \alpha^3\beta\gamma)$

[This can be obtained by performing the expansion, but only noting the types of term (some of which are repeated).]

[$1 + 3 + 1 + 3 = 8$ terms]

Thus $d = -(-5) - \sum \alpha^2\beta^2 - (-5)^2 - \alpha\beta\gamma \sum \alpha^2$

$= -20 - \sum \alpha^2\beta^2 - (-5) \sum \alpha^2$ (B)

So we need to find $\sum \alpha^2$, $\sum \alpha^2\beta^2$ & $\sum \alpha^2\beta$

First of all, consider $(\alpha+\beta+\gamma)^2 = \sum \alpha^2 + 2\sum \alpha\beta$,

so that $\sum \alpha^2 = 2^2 - 2(-4) = 12$

We can also consider $(\alpha\beta+\alpha\gamma+\beta\gamma)^2 = \sum(\alpha\beta)^2 + 2\sum \alpha^2\beta\gamma$

[giving $3 + 2(3) = 9$ terms]

so that $\sum \alpha^2\beta^2 = (-4)^2 - 2\alpha\beta\gamma\sum \alpha = 16 - 2(-5)(2) = 36$

Then $(\alpha+\beta+\gamma)(\alpha\beta+\alpha\gamma+\beta\gamma) = (\sum \alpha^2\beta) + 3\alpha\beta\gamma$

[$3(2) + 3 = 9$ terms]

so that $\sum \alpha^2\beta = 2(-4) - 3(-5) = 7$

Hence, from (A), $c = -14 + \sum \alpha^2\beta = -14 + 7 = -7$

and, from (B),

$d = -20 - \sum \alpha^2\beta^2 - (-5)\sum \alpha^2 = -20 - 36 + 5(12) = 4$

And so the required equation is $x^3 + 2x^2 - 7x + 4 = 0$

[In this example we can use the Factor theorem to see that $\alpha\ (say) = 1$, and that $\beta, \gamma = \frac{1 \pm \sqrt{21}}{2}$, which leads to $\alpha + \beta\gamma$ etc being -4, 1 & 1, enabling the new equation to be confirmed. In general of course, we may not be able to find a root by the Factor theorem.]