Probability Generating Functions (6 pages; 16/4/21)

(1) $G_X(s) = E(s^X) = \sum_{k=0}^{\infty} p_k s^k$

(2) $G_X(1) = \sum_{k=0}^{\infty} p_k = 1$

and $\sum_{k=0}^{\infty} p_k s^k \leq \sum_{k=0}^{\infty} p_k$ when $|s| \leq 1$, so that the series converges for $|s| \leq 1$

(3) Examples

(i) Bernouilli (single trial Binomial): $q + ps$

(ii) Binomial: $B(n, p)$; $p_k = \binom{n}{k} p^k q^{n-k}$

$G_X(s) = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} s^k = \sum_{k=0}^{n} \binom{n}{k} (ps)^k q^{n-k} = (q + ps)^n$

Notes

(a) $n = 1$ gives the Bernouilli distribution

(b) $G_X(s) = [G_Y(s)]^n$, where Y has the Bernouilli distribution

(generally true when $X = Y_1 + \cdots + Y_n$, where the Y_i have the same distribution)

(iii) Poisson: $P_o(\lambda)$; $p_k = \frac{e^{-\lambda}\lambda^k}{k!}$

$G_X(s) = \sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^k}{k!} s^k = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda s)^k}{k!} = e^{-\lambda}\left(e^{\lambda s}\right) = e^{\lambda(s-1)}$

(iv) Geometric: $p_k = q^{k-1}p$ (probability of 1st success on kth attempt)

$G_X(s) = \sum_{k=1}^{\infty} q^{k-1}ps^k = ps\sum_{k=1}^{\infty}(qs)^{k-1} = \frac{ps}{1-qs}$ if $|qs| < 1$; ie $|s| < \frac{1}{q}$

(v) Negative Binomial: $p_k = \binom{k-1}{n-1}p^{n-1}q^{(k-1)-(n-1)}p$

(probability of nth success on kth attempt

$= P(n-1\ successes\ in\ k-1\ trials) \times P(success\ on\ kth\ trial)$)

$= \binom{k-1}{n-1}p^n q^{k-n}$ ($k \geq n$)

$G_X(s) = \sum_{k=n}^{\infty}\binom{k-1}{n-1}p^n q^{k-n}s^k$

$= (ps)^n\sum_{k-n=0}^{\infty}\binom{k-1}{(k-1)-(n-1)}q^{k-n}s^{k-n}$

$= (ps)^n\sum_{k-n=0}^{\infty}\binom{k-1}{k-n}(qs)^{k-n}$

$= (ps)^n\sum_{r=0}^{\infty}\binom{n+r-1}{r}(qs)^r$

$= (ps)^n\{1 + nqs + \binom{n+1}{2}(qs)^2 + \binom{n+2}{3}(qs)^3 + \cdots\}$

$= (ps)^n\{1 + nqs + \frac{(n+1)n}{2!}(qs)^2 + \frac{(n+2)(n+1)n}{3!}(qs)^3 + \cdots\}$

$= (ps)^n\{1 + (-n)(-qs) + \frac{(-n)(-n-1)}{2!}(-qs)^2$

$+ \frac{-n(-n-1)(-n-2)n}{3!}(-qs)^3 + \cdots\}$

$= (ps)^n(1-qs)^{-n} = \left(\frac{ps}{1-qs}\right)^n$

Notes

(a) $n = 1$ gives the Geometric distribution

(b) $G_X(s) = [G_Y(s)]^n$, where Y has the Geometric distribution

(4) Uniqueness theorem:

$G_X(s) = G_Y(s)$ (for all s) $\Leftrightarrow P(X = k) = P(Y = k)$ for all k

ie X & Y have the same distribution

(5) Given $G_X(s)$, the p_k can be obtained by either of the following methods:

(a) expanding $G_X(s)$, to find the coefficient of s^k

(b) $p_k = \frac{1}{k!}G_X^{(k)}(0)$ (for $k > 0$)

(6) $G_X^{(r)}(1) = E[X(X-1)\ldots(X-[r-1])]$

$G_X'(1) = E[X]$

and $G_X''(1) = E[X(X-1)]$,

so that $Var(X) = E(X^2) - [E(X)]^2$

$+E[X(X-1)] + E[X] - [E(X)]^2$

$= G_X''(1) + G_X'(1) - [G_X'(1)]^2$

Example

If $X \sim P_o(\lambda)$, $G_X(s) = e^{\lambda(s-1)}$

$G'_X(s) = \lambda e^{\lambda(s-1)}$ & $G''_X(s) = \lambda^2 e^{\lambda(s-1)}$

$Var(X) = G''_X(1) + G'_X(1) - [G'_X(1)]^2$

$= \lambda^2 + \lambda - \lambda^2 = \lambda$

(7) If X & Y are independent random variables, then

$G_{X+Y}(s) = G_X(s)G_Y(s)$

Proof

$G_{X+Y}(s) = E(s^{X+Y}) = E(s^X s^Y)$

$= E(s^X)E(s^Y)$ (by independence)

$= G_X(s)G_Y(s)$

Example

If $X_1 \sim P_o(\lambda_1)$ & $X_2 \sim P_o(\lambda_2)$ and X_1 & X_2 are independent,

then $G_{X_1+X_2}(s) = G_{X_1}(s)G_{X_2}(s) = e^{\lambda_1(s-1)}e^{\lambda_2(s-1)} = e^{(\lambda_1+\lambda_2)(s-1)}$

$\Rightarrow X_1 + X_2 \sim P_o(\lambda_1 + \lambda_2)$

See STEP 2015, P3, Q12 for a variation on this.

(8) Let $Y = a + bX$. Then $G_Y(s) = E(s^Y) = s^a E(s^{bx}) = s^a G_X(s^b)$

(9) If $X_1, X_2, \ldots$ & N are independent random variables, where the X_i have pgf $G_X(s)$, then $S_N = X_1 + X_2 + \cdots + X_N$ has pgf

$G_{S_N}(s) = G_N\left(G_X(s)\right)$

Proof

$G_{S_N}(s) = E(s^{S_N}) = \sum_{n=0}^{\infty} E(\, s^{S_n})P(N = n)$

$= \sum_{n=0}^{\infty} E(s^{X_1} s^{X_2} \dots s^{X_n})\, P(N = n)$

$= \sum_{n=0}^{\infty} E(s^{X_1})E(s^{X_2}) \dots E(s^{X_n})\, P(N = n)$

(as the X_i are independent)

$= \sum_{n=0}^{\infty} \left(G_X(s)\right)^n P(N = n) = G_N\left(G_X(s)\right)$

(10) (With the same notation as in (9)) $E(S_N) = E(N)E(X)$

Proof

From (6), $E(S_N) = G'_{S_N}(1)$

and $G'_{S_N}(s) = \frac{d}{ds}[G_N\left(G_X(s)\right)]$, by (9)

$= G'_N\left(G_X(s)\right)G'_X(s)$

[noting that $G'_N\left(G_X(s)\right)$ means the derivative wrt $G_X(s)$]

So $E(S_N) = G'_{S_N}(1) = G'_N\left(G_X(1)\right)G'_X(1)$

$= G'_N(1)G'_X(1)$, as $G_X(1) = \sum_{k=0}^{\infty} p_k = 1$

$= E(N)E(X)$

(11) (With the same notation as in (9))

$Var(S_N) = E(N)Var(X) + Var(N)[E(X)]^2$

[See PGF Exercises for proof.]

(12) ['Poisson hen'] A hen lays N eggs, where $N \sim P_o(\lambda)$, and each egg has probability p of hatching. It can be shown that the total number of eggs that hatch $\sim P_o(\lambda p)$.

[See PGF Exercises for proof.]