Newton's Forward Difference method (9 pages; 12/6/20)

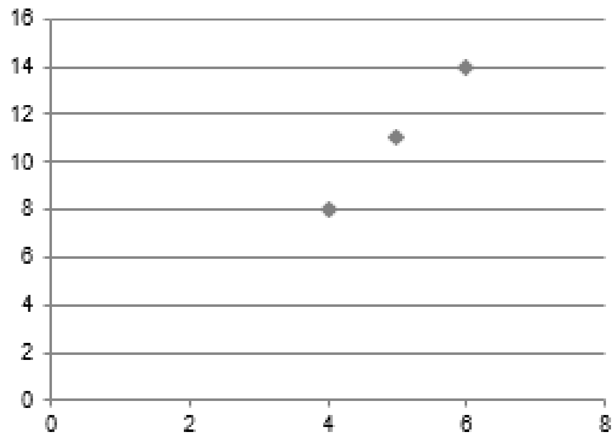
(1) This is one of two standard methods for approximating functions - the other being Lagrange's method.

aka Newton's forward difference interpolation, or Newton's interpolating polynomial

It is used to create a polynomial function that passes through given points. One use of the function is then to find interpolated values.

(2) A straight line can be drawn through any two points, and a quadratic curve through any three points that don't lie on a straight line. **In general, a polynomial function of order $n-1$ is needed if there are n points (unless the points happen to lie on a polynomial curve of order less than $n-1$).**

(3) Consider the 3 points (4,8), (5,11) & (6,14), which lie on a straight line.



A finite difference table can be constructed as follows:

x	f	Δf
4	8	
		3
5	11	
		3
6	14	

From this we obtain the straight line $f(x) = 8 + 3(x - 4)$

(4) If instead there is a gap of 2 in the x values; eg if the points are (4,8), (6,14) & (8,20), then we have the following finite difference table:

x	f	Δf
4	8	
		6
6	14	
		6
8	20	

and this gives the straight line $f(x) = 8 + \frac{6}{(6-4)}(x - 4)$,

which in the more general case becomes $f(x) = f_0 + \frac{\Delta f_0}{h}(x - x_0)$

Note that $\frac{\Delta f_0}{h} = \frac{f_1 - f_0}{x_1 - x_0}$ is the gradient.

(5) For a larger number of points, this extends to:

$$f(x) = f_0 + \frac{x-x_0}{h}\Delta f_0 + \frac{(x-x_0)(x-x_1)}{2!h^2}\Delta^2 f_0$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)}{3!h^3}\Delta^3 f_0 + \ \ldots$$

where $\Delta f_i = f_{i+1} - f_i$ and $\Delta^2 f_i = \Delta f_{i+1} - \Delta f_i$,

$h = x_1 - x_0$, and the x values have to be evenly spaced

(6) In the simple case of fitting a straight line to the points (x_0, f_0) & (x_1, f_1), Newton's Forward Difference method gives

$f(x) = f_0 + \frac{x-x_0}{h}\Delta f_0$, which can be rewritten as

$\frac{f(x)-f_0}{x-x_0} = \frac{\Delta f_0}{h}$ or $\frac{f(x)-f_0}{x-x_0} = \frac{f_1-f_0}{x_1-x_0}$; ie equating two expressions for the gradient of the line (noting that $(x, f(x))$ is a general point on the line).

(7) If (for example) the 3rd differences ($\Delta^3 f_0$) are constant, then the 4th and subsequent differences will be zero, and the polynomial will be of order 3 (ie a cubic).

If the 3rd differences are approximately constant, then a cubic function may be a reasonably good fit.

(8) Regardless of whether the 3rd differences are constant, if the starting point for Newton's Forward Difference method is x_0, then if a cubic is fitted, it will pass through the points (x_0, f_0), (x_1, f_1), (x_2, f_2) & (x_3, f_3).

[Note that x_3 doesn't appear explicitly in the cubic polynomial, though $\Delta^3 f_0 = \Delta^2 f_1 - \Delta^2 f_0 = (\Delta f_2 - \Delta f_1) - (\Delta f_1 - \Delta f_0)$

$= \Delta f_2 - 2\Delta f_1 + \Delta f_0$

$= (f_3 - f_2) - 2(f_2 - f_1) + (f_1 - f_0)$

$= f_3 - 3f_2 + 3f_1 - f_0]$

If necessary, the starting point can be some other point (if, for example, interpolation beyond x_0 is required).

(9) Exercise: In the case where a quadratic is fitted to the points (x_0, f_0), (x_1, f_1), (x_2, f_2), confirm that $f(x_2) = f_2$

Solution

$f(x_2) = f_0 + \frac{x_2 - x_0}{h}\Delta f_0 + \frac{(x_2 - x_0)(x_2 - x_1)}{2!h^2}\Delta^2 f_0$

$\Delta f_0 = f_1 - f_0$ and $\Delta^2 f_0 = \Delta f_1 - \Delta f_0 = (f_2 - f_1) - (f_1 - f_0)$

so that $f(x_2) = f_0 + \frac{2h}{h}(f_1 - f_0) + \frac{(2h)h}{2h^2}\{(f_2 - f_1) - (f_1 - f_0)\}$

$f_0 + 2(f_1 - f_0) + \{(f_2 - f_1) - (f_1 - f_0)\} = f_2$

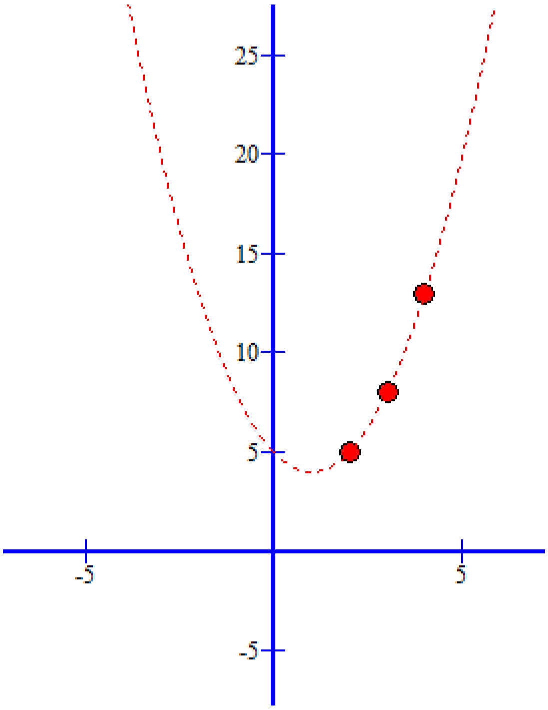
(10) Exercise: If $f(x)$ passes through the points (2,5), (3,8) & (4,13):

(i) Find the quadratic function obtained from Newton's interpolating polynomial.

(ii) Estimate $f(2.5)$

Solution

(i)



x	f	Δf	$\Delta^2 f$
2	5		
		3	
3	8		2
		5	
4	13		

$$f(x) = f_0 + \frac{x-x_0}{h}\Delta f_0 + \frac{(x-x_0)(x-x_1)}{2!h^2}\Delta^2 f_0$$

becomes $f(x) = 5 + \frac{(x-2)}{1}(3) + \frac{(x-2)(x-3)}{2}(2)$

$= 5 - 6 + 6 + x(3-5) + x^2 = x^2 - 2x + 5$

$y = x^2 - 2x + 5$

(ii) From $f(x) = 5 + \frac{(x-2)}{1}(3) + \frac{(x-2)(x-3)}{2}(2)$, we obtain

$f(2.5) = 5 + \frac{(2.5-2)}{1}(3) + \frac{(2.5-2)(2.5-3)}{2}(2)$

$= 5 + \frac{3}{2} - \frac{1}{4} = \frac{25}{4} = 6.25$

(ie there is no need to simplify the quadratic to $y = x^2 - 2x + 5$ if only the interpolated value is required)

(11) Suppose that we are given the following points

(3,6) , (5,9) , (7,14) , (9,17) , (11,18) , (13,16) , (15,13)

The (evenly spaced) x values would be indicated by

$x = 3(2)15$

The finite difference table for these points is:

x	f	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$	$\Delta^5 f$	$\Delta^6 f$
3	6						
		3					
5	9		2				
		5		-4			
7	14		-2		4		
		3		0		-5	
9	17		-2		-1		9
		1		-1		4	
11	18		-3		3		
		-2		2			
13	16		-1				
		-3					
15	13						

(12) Exercise: (i) Use Newton's interpolating polynomial to fit a cubic function to the points:

(3,6) , (5,9) , (7,14) , (9,17) , (11,18) , (13,16) , (15,13)

[the same points that were used earlier on]

(ii) Obtain an estimate for $f(6)$.

Solution

From the earlier finite difference table,

$x = 3(2)15;\ \ f_0 = 6,\ \ \Delta f_0 = 3,\ \ \Delta^2 f_0 = 2, \Delta^3 f_0 = -4$

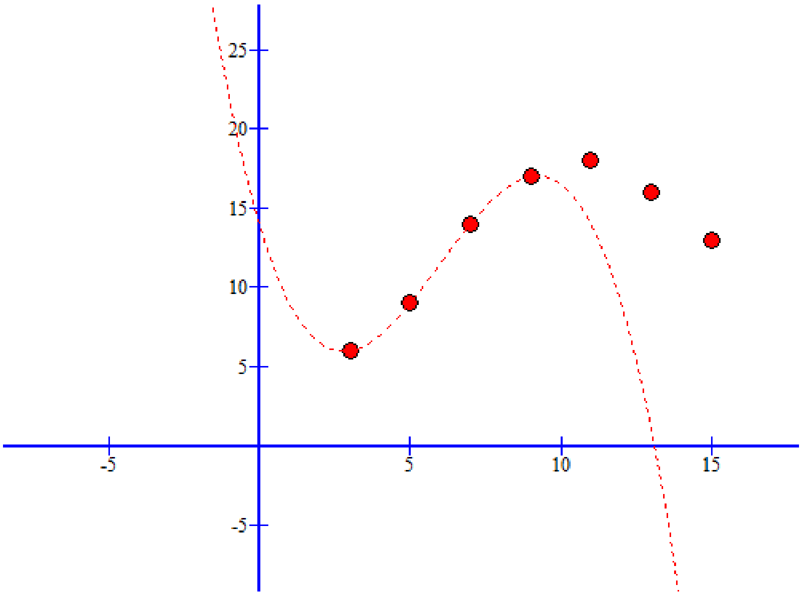
So $f(x) = f_0 + \frac{x-x_0}{h}\Delta f_0 + \frac{(x-x_0)(x-x_1)}{2!h^2}\Delta^2 f_0$

$+ \frac{(x-x_0)(x-x_1)(x-x_2)}{3!h^3}\Delta^3 f_0 + \ \ \ldots$

and $f(x) \approx 6 + \frac{(x-3)}{2}(3) + \frac{(x-3)(x-5)}{2(4)}(2) + \frac{(x-3)(x-5)(x-7)}{6(8)}(-4)$

$= 6 + \frac{3(x-3)}{2} + \frac{(x-3)(x-5)}{4} - \frac{(x-3)(x-5)(x-7)}{12}$

which can be simplified to $f(x) = -\frac{1}{12}x^3 + \frac{3}{2}x^2 - \frac{77}{12}x + 14$



Note that the 3rd difference is not approximately constant, and so a cubic function is not a very good fit overall; only for the points close to x_0.

(ii) From the unsimplified polynomial,

$f(6) \approx 6 + 4.5 + 0.75 + 0.25 = 11.5$

(13) The interpolating polynomial need not be taken about x_0.

e.g. x_3 could be used instead:

$$f(x) = f_3 + \frac{x-x_3}{h}\Delta f_3 + \frac{(x-x_3)(x-x_4)}{2!h^2}\Delta^2 f_3 +$$

$$\frac{(x-x_3)(x-x_4)(x-x_5)}{3!h^3}\Delta^3 f_3 + \ \ldots$$

The polynomial will then pass through (x_3, f_3).

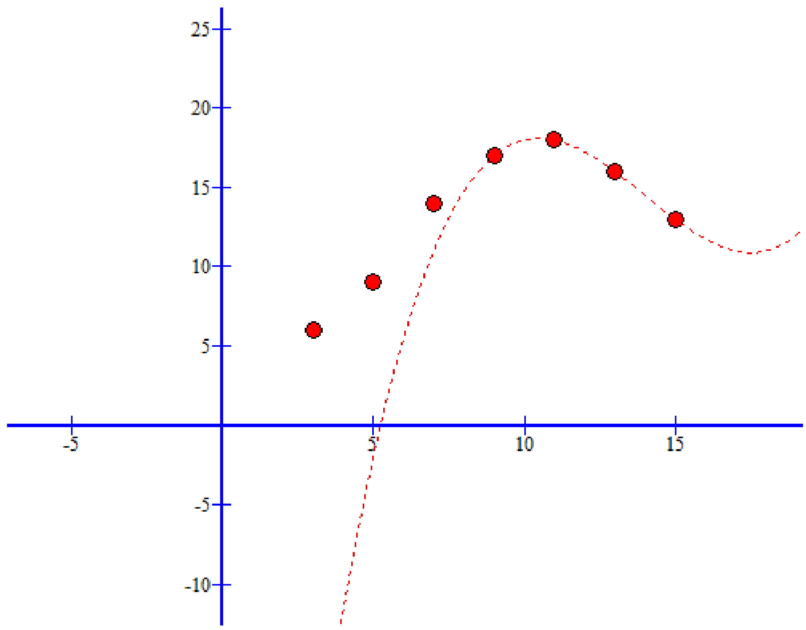
From the earlier finite difference table,

$x = 9(2)15,\ f_3 = 17,\ \ \Delta f_3 = 1,\ \Delta^2 f_3 = -3, \Delta^3 f_0 = 2$

and $f(x) \approx 17 + \frac{(x-9)}{2}(1) + \frac{(x-9)(x-11)}{2(4)}(-3)$

$+ \frac{(x-9)(x-11)(x-13)}{6(8)}(2)$

$= \frac{1}{24}x^3 - \frac{7}{4}x^2 + \frac{551}{24}x - \frac{313}{4}$



(14) Exercise: Confirm the standard result for a quadratic sequence $an^2 + bn + c$ that a is half the (constant) 2nd difference.

Solution

As an example (changing n to x), consider the quadratic function

$f(x) = 6x^2 - 5x + 1$

Finite Difference table:

x	f	Δf	$\Delta^2 f$
1	2		
		13	
2	15		12
		25	
3	40		12
		37	
4	77		12
		49	
5	126		12
		61	
6	187		12
		73	
7	260		

$f(x) = f_0 + \frac{x-x_0}{h}\Delta f_0 + \frac{(x-x_0)(x-x_1)}{2!h^2}\Delta^2 f_0$

When $h = 1$, the coefficient of x^2 is $\frac{\Delta^2 f_0}{2}$