Network Flows - Introduction (5 pages; 16/1/21)

(1) Diagram 1 below shows a network of pipes.

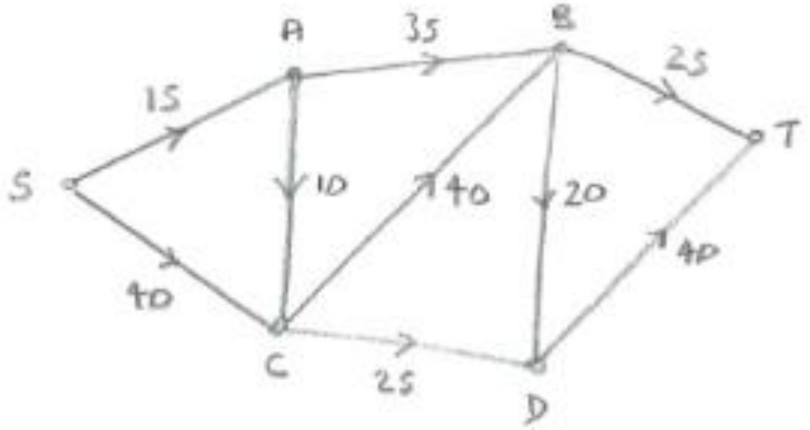

Diagram 1

The weights on the arcs are the maximum flows that are possible in the individual pipes (in a given period of time – so effectively we are considering rates of flow).

We can attempt to find the maximum overall flow through the network, as follows:

Start with a flow of 15 along SABT.

Then add a flow of 25 along SCDT.

This allows a further flow of 10 to be added along SCBT (as the maximum allowable flow along BT is 25, and only 15 has been used so far).

This gives a total flow of 50.

Now the maximum overall flow through the network cannot exceed the sum of the maximum flows along SA and SC; ie

$15 + 40 = 55$.

We can in fact find another flow of 5: along SCBDT.

And so the maximum overall flow through the network is:

15 (SABT) + 25 (SCDT) + 10 (SCBT) + 5 (SCBDT) = 55

The flows in the individual arcs are shown below:

	SABT	SCDT	SCBT	SCBDT	Total	Max. allowed
SA	15				15	15
SC		25	10	5	40	40
AB	15				15	35
AC					0	10
BD				5	5	20
BT	15		10		25	25
CB			10	5	15	40
CD		25			25	25
DT		25		5	30	40

From this we can construct the network of actual flows, as shown in diagram 2.

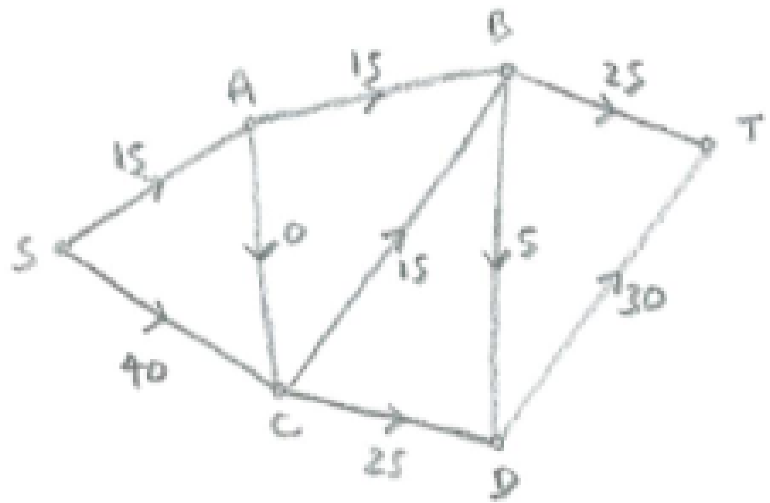

Diagram 2

Notes

(i) In the context of network flows, the term ‘capacity’ tends to be used, instead of ‘weight’ - although this can be confusing.

Thus the weights on the arcs in Diagram 1 may be referred to as ‘capacities’ - though ‘upper capacities’ is also used (and is clearer). The weights on the arcs in Diagram 2 may also be referred to as ‘capacities’ (‘actual flows’ would probably be better).

(ii) The starting node is termed the ‘source’ (usually labelled as S), whilst the end node is termed the ‘sink’ (usually labelled as T). There may be more than one source, and more than one sink. [In this case, a ‘super-source’ node can be created, and joined to each of the sources, with large upper capacities on the new arcs (so that they don’t place any constraints on the maximum overall flow – alternatively, these upper capacities can just be set equal to the largest of the sums of the upper capacities leading from each source). ‘Super-sinks’ are then created in the same way.]

(iii) The arcs of the network need not all be directed (ie it may be possible for a flow to occur either way along a particular arc). However, arcs leading from the source node, or to the sink node will always be directed.

(iv) In a network of actual flows, it will always be the case that the total flow into each node will equal the total flow out of the node (ie no build up is allowed at a node).

(2) ‘Maximum flow - minimum cut’ theorem

A ‘cut’ can be thought of as a border that runs across the network, from top to bottom. (Imagine a network of roads spread across two countries.)

Cuts can be defined by placing nodes into two groups: those to the left of the cut (including S) and those to the right (including T).

Diagram 3 shows the cuts {SAC}{BDT} and {SACD}{BT}.

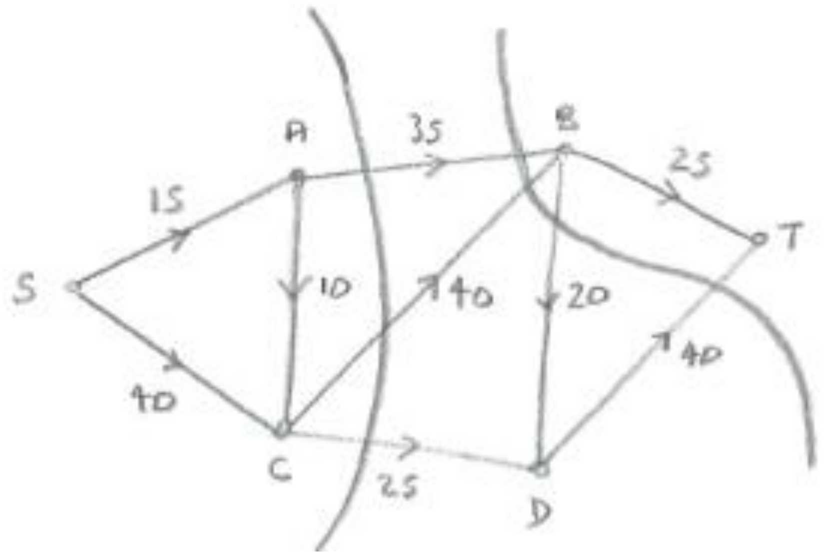

Diagram 3

Each cut can be given a value, representing the maximum possible flow across the cut, from left to right (so flows from right to left are not counted).

Thus the value for {SACD}{BT} is $35 + 40 + 40 = 115$ (and note that the flow of 20 for BD is excluded, as it is a flow from right to left). [Note that it is the network of maximum capacities that is used to find the values of the cuts – not the network of actual flows.]

For diagram 1, the possible cuts, together with their values are:

{S}{ABCDT} $15 + 40 = 55$

{SA}{BCDT} $35 + 10 + 40 = 85$

{SC}{ABDT} $15 + 40 + 25 = 80$

{SAB}{CDT} $25 + 20 + 10 + 40 = 95$

{SCD}{ABT} $15 + 40 + 40 = 95$

{SAC}{BDT} $35 + 40 + 25 = 100$

{SABC}{DT} $25 + 20 + 25 = 70$

{SACD}{BT} $35 + 40 + 40 = 115$

{SABCD}{T} $25 + 40 = 65$

Each cut places a constraint on the maximum overall flow through the network: it cannot exceed the value of the cut. (For a network of roads that is cut by a national border, the value of the cut places an upper limit on the flow (in a given time) across the border.)

So, in the above example, the maximum overall flow through the network cannot exceed the smallest of the values of the cuts; ie 55 for {S}{ABCDT}.

The 'Maximum flow - minimum cut' theorem says that this upper limit for the overall flow through the network can always be attained.

However, it doesn't provide a method for finding a network of actual flows that achieves it. (There is a separate algorithm for this though.)

Also, as seen, it can be time-consuming to list all the possible cuts, in order to find the one with the smallest value.

But if an overall flow through the network can be found that equals the value of a particular cut, then this cut must be a minimum one (if there were a smaller one, it would be less than than the flow found, and this contradicts the 'Maximum flow - minimum cut' theorem), and the flow found must therefore be the maximum one (by the 'Maximum flow - minimum cut' theorem).

In the above example, the flow found had a value of 55, and the fact that this equals the value of the cut {S}{ABCDT} means that 55 is the maximum flow.