Impulse, Momentum & Coefficient of Restitution

(14 pages; 19/2/17)

(1) Momentum of an object is defined as: mass × velocity (mv)

As velocity is a vector, momentum is as well.

Units: Ns (as $N = kgms^{-2}$, from Newton's 2nd Law: $F = ma$)

(2) Newton's 2nd Law was originally stated in terms of momentum:

"Force = Rate of change of momentum",

since $\frac{d}{dt}(mv) = m\frac{dv}{dt} = ma$

(3) The impulse of a force F acting over a time t is defined as Ft

Units: Ns

Common symbol: J (I is used for moment of inertia)

(4) Example: Hammer hitting a nail.

In many situations, the force is large and acting for a very short period of time. It often isn't feasible to establish the size of either the force or the time.

The force could vary over time, in which case the total impulse would be determined by integration: $lim \sum F(t).\delta t = \int F(t)\, dt$, in the theoretical case where the force is a known function of time.

(5) In the case where the force (and hence the acceleration) is assumed to be constant, the suvat equation $v = u + at$

$\Rightarrow F = ma = m\frac{v-u}{t} \Rightarrow Ft = mv - mu$

This can be shown to be true in all cases (ie whether the force is constant or not).

Thus, Impulse = change in momentum (Impulse - momentum principle)

(6) **Example**: A car has mass 1 tonne ($1000kg$) and a maximum driving force of $500N$. Find the minimum time taken to increase its speed from $40kmph$ to 50 $kmph$.

Solution

$Ft = mv - mu$

$1\ kmph = \frac{1000}{3600} = \frac{5}{18} ms^{-1}$

$\Rightarrow 500t = 1000(50 - 40)\left(\frac{5}{18}\right)$

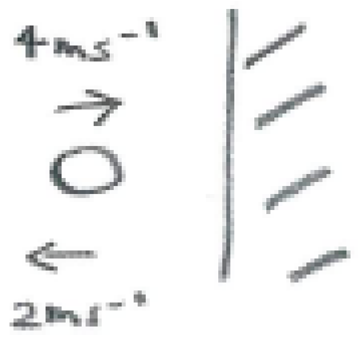
$\Rightarrow t = 5.56\ s$

(7) Conservation of momentum

If no external forces are acting, then the total momentum of a system will remain constant.

By Newton's 3rd law, the internal forces balance (and hence the impulses associated with them), so that the net impulse on the system is zero, and the change in momentum is therefore also zero.

(8) **Example** : Ball hitting a wall



Ball of mass $0.2kg$

Impulse of wall on ball = change in momentum of ball

$= 0.2(2 - (-4)) = 1.2\ Ns$

(taking right to left as the positive direction)

(9) **Example** : Bat hitting ball

A cricket ball of mass $0.16kg$ travels with velocity $30\underline{i}\ ms^{-1}$ towards a batsman, who hits it with an impulse of $-8\underline{i} + 2\underline{j}\ Ns$

What is the speed and direction of the ball after being hit?

Solution

$[30ms^{-1} \approx 30 \times \frac{3600}{1600} = 30 \times 2.25 = 67.5\ mph]$

Impulse = Change in momentum ⇒

$\begin{pmatrix}-8\\2\end{pmatrix} = 0.16\begin{pmatrix}v_x\\v_y\end{pmatrix} - 0.16\begin{pmatrix}30\\0\end{pmatrix}$

$\Rightarrow \ldots \begin{pmatrix}v_x\\v_y\end{pmatrix} = \begin{pmatrix}-20\\12.5\end{pmatrix}$

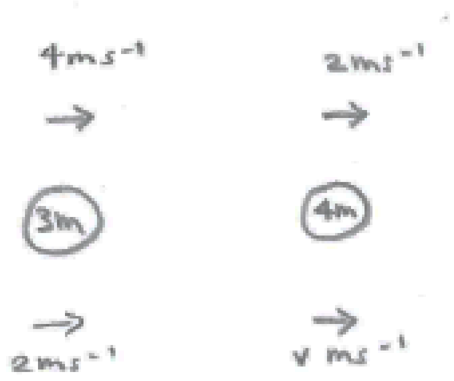
Hence speed of ball after being hit = $\sqrt{(-20)^2 + (12.5)^2}$

$= 23.6\ ms^{-1}$ (3sf)

The ball makes an angle of $tan^{-1}\left(\frac{12.5}{20}\right) = 32.0^0$ (3sf) to the line joining the wickets.

(10) Collision Examples

Example 1



(The top and bottom rows show the 'before' and 'after' velocities.)

Conservation of momentum ⇒

$3m(4) + 4m(2) = 3m(2) + 4mv$

$\Rightarrow 20 = 6 + 4v \Rightarrow v = 3.5$

Impulse on $3m$ ball

$= 3m(2 - 4) = -6m$ to the right

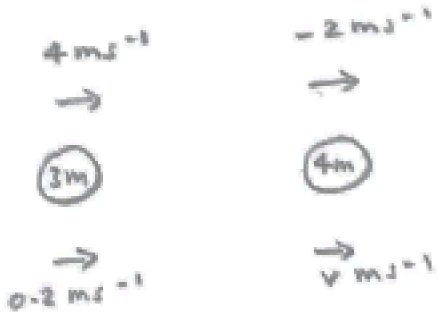
$= 6m\ Ns$ to the left

Impulse on $4m$ ball

$= 4m(3.5 - 2)$ to the right

$= 6m\ Ns$ to the right

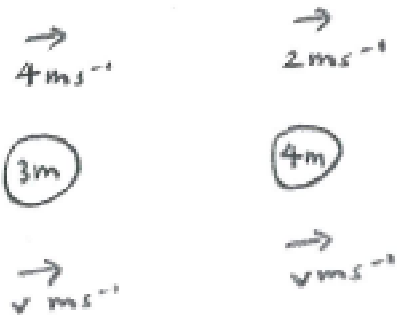
Example 2



CoM $\Rightarrow 3m(4) + 4m(-2) = 3m(0.2) + 4mv$

$\Rightarrow 4 = 0.6 + 4v \Rightarrow v = 0.85$

Example 3

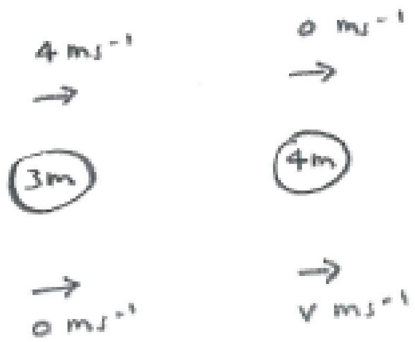


Here the balls are coalescing (ie sticking together).

CoM $\Rightarrow 3m(4) + 4m(2) = (3m + 4m)v$

$\Rightarrow 20 = 7v \Rightarrow v = 2.86$ (3sf)

Example 4



CoM $\Rightarrow 3m(4) + 4m(0) = 3m(0) + 4mv$

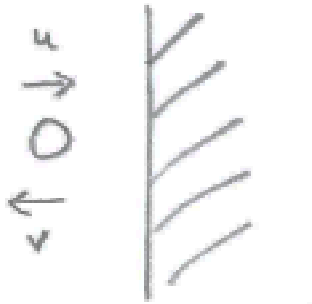
$\Rightarrow 12 = 4v \Rightarrow v = 3$

Notes

(i) It is recommended to draw all arrows from left to right (with negative values, as necessary).

(ii) With the positive direction being from left to right, the final velocity of the right-hand ball should be greater than its initial velocity (as it is receiving an impulse from the left), and it should also be greater than the final velocity of the left-hand ball (in order for the right-hand ball to move ahead of the left-hand ball after the collision).

(11) Newton's Law of Restitution



If the incoming speed is u and the outgoing speed is v , then the coefficient of restitution (e) is defined by:

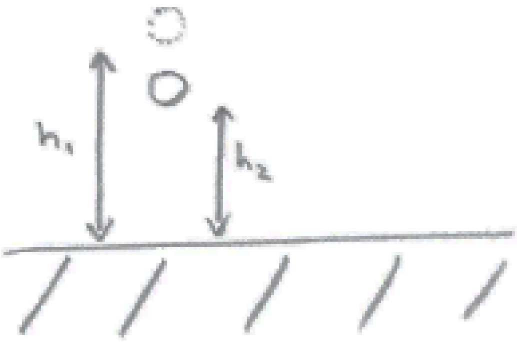
$$e = \frac{v}{u}$$

e^2 is then the factor by which the kinetic energy of the ball is reduced:

$$e^2 = \frac{\frac{1}{2}mv^2}{\frac{1}{2}mu^2}$$

The coefficient of restitution depends on both of the objects involved in a collision (in this case, the ball and the wall), and is a measure of the 'bounciness' of the two objects.

(12)



If a ball with coefficient of restitution e (with the given surface) is dropped from a height h_1 and bounces to height h_2, then

$$e^2 = \frac{\frac{1}{2}mv^2}{\frac{1}{2}mu^2} = \frac{mgh_2}{mgh_1} \text{ (by Conservation of Mechanical Energy)}$$

$$= \frac{h_2}{h_1}$$

(13) Range of values for e

By its definition, $0 \leq e \leq 1$

$e = 0$ is possible: in this case the objects coalesce, and the collision is described as 'perfectly inelastic' (or 'perfectly plastic' - as in plasticine)

$e = 1$ however is only a theoretical possibility, referred to as 'perfect elasticity', where all the kinetic energy possessed by each object is converted into elastic potential energy, which is then converted back into the original amount of kinetic energy.

Example values of e:

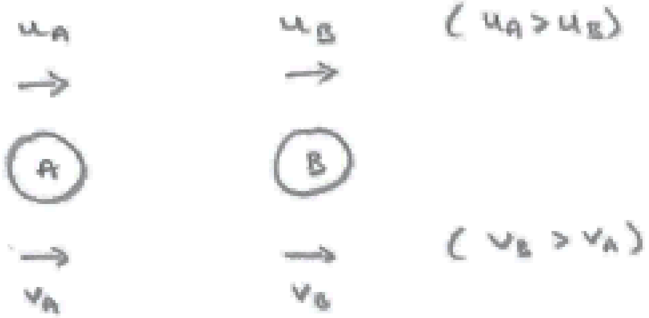
table tennis ball: 0.9

golf ball : 0.83 (maximum allowed)

basketball : 0.6

These values are arrived at by bouncing on a specified surface; often steel.

(14) Two ball collisions



The ball & wall situation can be extended to one with two balls by considering the relative velocities:

incoming speed becomes 'speed of approach' : $u \longrightarrow u_A - u_B$

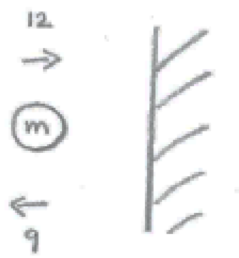
outgoing speed becomes 'speed of separation' : $v \longrightarrow v_B - v_A$
(note the reversal of A & B)

$e = \frac{v_B - v_A}{u_A - u_B}$

(Newton's Law of Restitution – aka Newton's Law of Impact or Newton's Experimental Law)

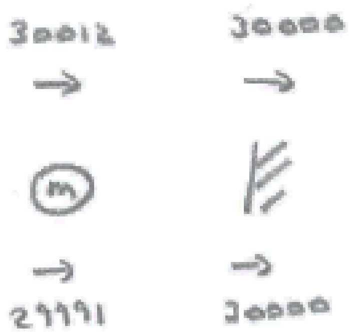
(15) Justification of two-ball formula

(a) ball hitting (stationary) wall: $e = \frac{9}{12}$



(b) Now consider the speeds relative to the Sun (as in the diagram below), where the units are ms^{-1}.

$e = \frac{30000-29991}{30012-30000} = \frac{9}{12}$

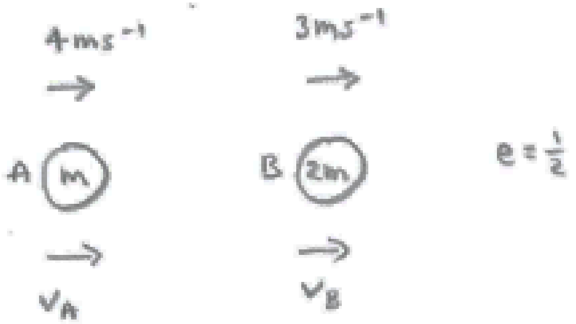


(c) This can be extended to the general definition:

$e = \frac{v_B - v_A}{u_A - u_B}$

(16) Examples

Example 1

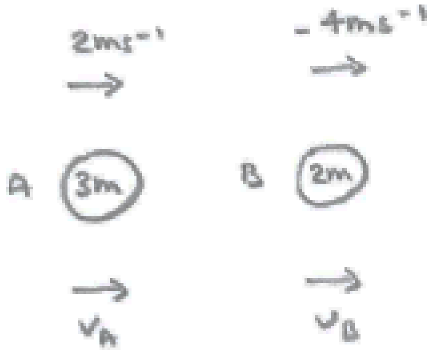


CoM $\Rightarrow 4m + 3(2m) = mv_A + 2mv_B \Rightarrow 10 = v_A + 2v_B$ (1)

NLR $\Rightarrow \frac{v_B - v_A}{4-3} = \frac{1}{2} \quad \Rightarrow v_B - v_A = \frac{1}{2}$ (2)

$\Rightarrow \ldots\ v_A = 3\ ms^{-1}$ & $v_B = 3.5\ ms^{-1}$

Example 2



where $e = 1$

CoM $\Rightarrow 2(3m) + (-4)(2m)$

$= 3mv_A + 2mv_B$

$\Rightarrow -2 = 3v_A + 2v_B \;\; (1)$

NLR $\Rightarrow \frac{v_B - v_A}{2-(-4)} = 1 \quad \Rightarrow \; v_B - v_A = 6 \; (2)$

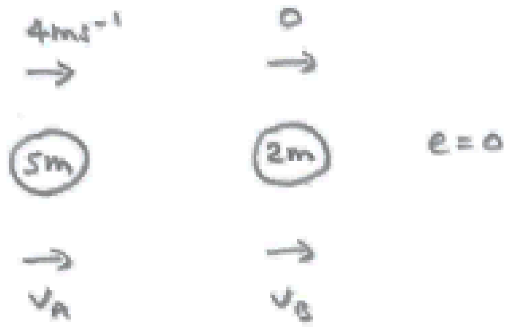
Subst. for v_B from (2) into (1) $\Rightarrow -2 = 3v_A + 2(6 + v_A)$

$\Rightarrow v_A = -\frac{14}{5} = -2.8\, ms^{-1}$ & $v_B = 3.2ms^{-1}$

Original kinetic energy $= \frac{1}{2}(3m)(2)^2 + \frac{1}{2}(2m)(-4)^2 = 22m\, J$

Final kinetic energy $= \frac{1}{2}(3m)(-2.8)^2 + \frac{1}{2}(2m)(3.2)^2 = 22m\, J$

Example 3



CoM $\Rightarrow 4(5m) + (0)(2m)$

$= 5mv_A + 2mv_B$

$\Rightarrow 20 = 5v_A + 2v_B$ (1)

NLR $\Rightarrow \frac{v_B - v_A}{4 - 0} = 0 \quad \Rightarrow \; v_B - v_A = 0$ (2)

$\Rightarrow v_A = v_B = \frac{20}{7} = 2.8571 = 2.86\ ms^{-1}$ (3sf)

Loss of KE $= \frac{1}{2}(5m)(4)^2 + 0 - \frac{1}{2}(7m)(2.8571)^2 = 11.4m\,J$ (3sf)

(17) Conservation of energy

For two balls colliding directly on a smooth surface, we can show that kinetic energy is always conserved when $e = 1$.

Let the two balls have masses m_A & m_B, initial speeds u_A & u_B and final speeds v_A & v_B (where the speeds are from left to right, and $u_A > 0$, with $u_A > u_B$).

Then, by conservation of momentum,

$m_A u_A + m_B u_B = m_A v_A + m_B v_B$ (1)

and, by Newton's law of impact, $\frac{v_B - v_A}{u_A - u_B} = e = 1$ (2)

Result to prove: $\frac{1}{2}m_A({v_A}^2 - {u_A}^2) + \frac{1}{2}m_B({v_B}^2 - {u_B}^2) = 0$ (3)

From (1), $m_B(v_B - u_B) = m_A(u_A - v_A)$,

and from (2), $(v_B + u_B) = (u_A + v_A)$.

Then, substituting into (3),

$LHS = \frac{1}{2}m_A(v_A - u_A)(v_A + u_A) + \frac{1}{2}m_B(v_B - u_B)(v_B + u_B)$

$= \frac{1}{2}m_A(v_A - u_A)(v_A + u_A) + \frac{1}{2}m_A(u_A - v_A)(u_A + v_A) = 0$,

as required.

[Note: From Example 3 above, it is clearly not the case that all kinetic energy is lost when $e = 0$]

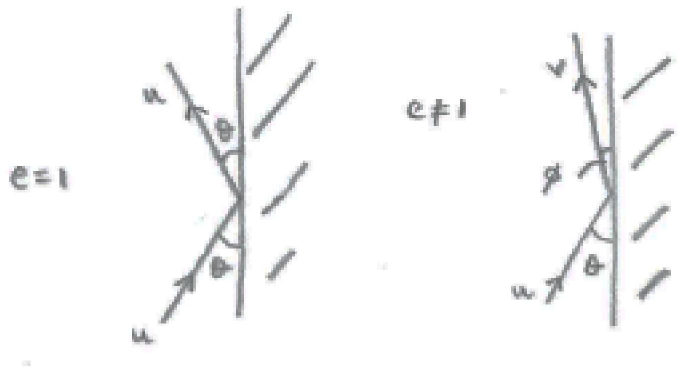
(18) Notes

(i) It might seem strange that there is no reference to mass in the relation e $= \frac{v_s}{v_a}$. However, it is involved indirectly, since v_s will be determined by conservation of momentum.

(ii) Whilst energy is lost (as heat & sound), momentum is conserved because the net change in momentum = net impulse = 0, since A and B exert equal and opposite impulses, by Newton's 3rd Law ($F_A = -F_B \Rightarrow F_A t = -F_B$t).

(19) Oblique impact with smooth plane

The diagram below could represent, for example, a ball hitting the cushion on a snooker table.



There is no impulse parallel to the plane (as there is no friction).

Resolve the velocity into components perpendicular to and parallel to the plane:

- component parallel to the plane is unchanged

- apply NLR to the perpendicular component

Thus: $vcos\phi = ucos\theta$; $vsin\phi = eusin\theta$

$\Rightarrow tan\phi = etan\theta$

eg $\theta = 30°, e = \frac{1}{2} \Rightarrow \ \phi = \arctan\left(\frac{1}{2} . \frac{1}{\sqrt{3}}\right) = 16.1°$