Method of Differences (3 pages; 6/10/20)

(1) **Example:** Given that $\frac{1}{r+2} - \frac{1}{r+3} = \frac{1}{r^2+5r+6}$,

$$\sum_{r=1}^{n} \frac{1}{r^2+5r+6} = \sum_{r=1}^{n} \frac{1}{r+2} - \sum_{r=1}^{n} \frac{1}{r+3}$$

$$= \frac{1}{3} + [\frac{1}{4} + \frac{1}{5} \ldots + \frac{1}{n+2}]$$

$$- \left([\frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{n+2}] + \frac{1}{n+3}\right)$$

As the terms in square brackets cancel:

$$= \frac{1}{3} - \frac{1}{n+3}$$

$$= \frac{(n+3)-3}{3(n+3)} = \frac{n}{3(n+3)}$$

(2) **Example**: $\sum_{r=1}^{n}\{\frac{1}{2(r+1)} - \frac{1}{(r+2)} + \frac{1}{2(r+3)}\}$

$$= \frac{1}{2}\sum_{r=1}^{n}\{\frac{1}{(r+1)} - \frac{2}{(r+2)} + \frac{1}{(r+3)}\}$$

$$= \frac{1}{2}S,$$

where $S = (\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{n} + \frac{1}{n+1})$

$$-2(\frac{1}{3} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2})$$

$$+(\frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3})$$

Highlighted items cancel,

so that the required sum $= \frac{1}{2}\left\{\frac{5}{6} - \frac{2}{3} - \frac{1}{n+2} + \frac{1}{n+3}\right\}$

$= \frac{1}{12} - \frac{1}{2(n+2)} + \frac{1}{2(n+3)}$

(3) Notes

(i) A handwritten version would have a box round the highlighted items.

(ii) For exam purposes, a typical requirement would be to see at least two items cancelling at the start (where the first item in the previous Example is $\frac{1}{4} - \frac{2}{4} + \frac{1}{4}$), and perhaps one at the end (though two would be best, to be on the safe side). Leave plenty of space for extra terms to be inserted.

(iii) The layout in the previous Example is clearer than:

$$\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{2}{4} + \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{2}{5} + \frac{1}{6}\right) + \left(\frac{1}{5} - \frac{2}{6} + \frac{1}{7}\right) + \cdots$$

(iv) In the previous Example, it probably isn't worth simplifying to a single fraction (unless asked to do so).

Had the answer been $\frac{1}{2(n+2)} - \frac{1}{2(n+3)}$, then

the form $\frac{(n+3)-(n+2)}{2(n+2)(n+3)} = \frac{1}{2(n+2)(n+3)}$ would usually be preferable.

(v) A check can be made by substituting $n = 1$:

$$\frac{1}{2(1+1)} - \frac{1}{(1+2)} + \frac{1}{2(1+3)} = \frac{1}{4} - \frac{1}{3} + \frac{1}{8} = \frac{6-8+3}{24} = \frac{1}{24}$$

and $\frac{1}{12} - \frac{1}{2(1+2)} + \frac{1}{2(1+3)} = \frac{1}{12} - \frac{1}{6} + \frac{1}{8} = \frac{2-4+3}{24} = \frac{1}{24}$ also

(vi) If asked to show that $\frac{1-r}{(r+1)(r+2)} = \frac{2}{r+1} - \frac{3}{r+2}$, for example, there is no need to use Partial Fractions; instead, just rearrange the RHS to obtain the LHS.

(vii) Beware of writing $\frac{1}{r+1} - \frac{1}{r+2} = \frac{r+2-r+1}{(r+1)(r+2)}$, when you mean $\frac{r+2-(r+1)}{(r+1)(r+2)}$.