Mechanics - Important Ideas: Equilibrium

(12 pages; 21/7/23)

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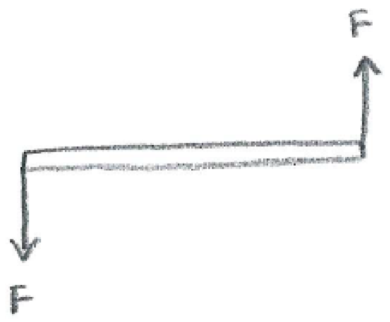
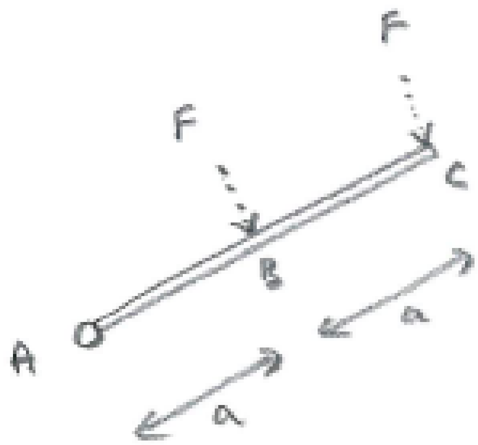
(1) When particle model isn't appropriate

In many Mechanics models, an object is treated as a particle. This means that any rotation of the object is not considered. In some cases this isn't appropriate. For example, in the diagram below, the forces on the object balance, but clearly the object is not in 'rotational equilibrium': overall, the forces have a turning effect.



(2) Definition of moment of force

In the diagram below, AC represents a door, with its hinge at A. Clearly a force F applied at C will have a greater turning effect than the same force applied at B.



The turning effect of the force at C (its 'moment about A') is defined to be $-F(2a)$ (as anti-clockwise moments are deemed to be positive).

More generally, the moment of a force $\underline{F}$ about a point A is defined to be $|\underline{F}| \times$ the shortest distance of the line of action of $\underline{F}$ from A, with a negative sign if the turning is in a clockwise sense.

[Note: The vector specification of a force (ie its magnitude and direction) is not sufficient: we also need to know the line along which it acts. This can be determined from a particular point where the force acts (together with the direction of the force).]

The unit of a moment of a force is Nm.

(3) Rotational equilibrium

(3.1) Most situations in which moments are used (at A level) concern stationary 'rigid bodies' (objects that maintain their shape under the action of forces), which are therefore in equilibrium.

In addition to resolving forces in two perpendicular directions, and applying Newton's 2nd law, we can also use the fact that there is rotational equilibrium to say that the net moment (of all the forces on the object) is zero; ie there is no net turning effect.

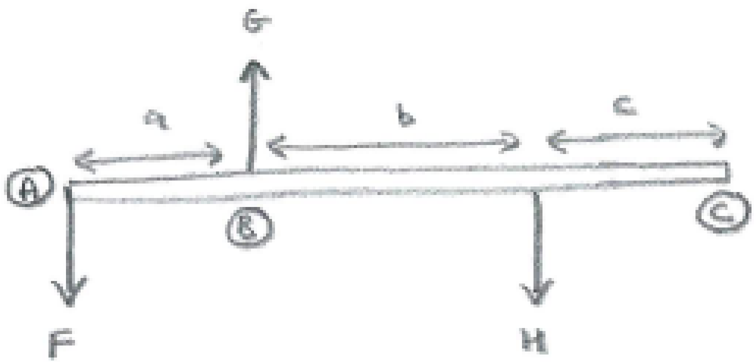
The question remains as to which point to take moments about.

It will be shown next that, provided the forces are in equilibrium, it doesn't matter which point we choose. Also, the point needn't actually be within the object itself (though it usually is).

(3.2) In the case of 3 forces (with no forces being parallel), the lines of action must meet at a point (be 'concurrent') in order for there to be rotational equilibrium. [Were one of the lines of action to not pass through the intersection of the other two, then there would be a non-zero moment about the point of intersection; meaning that there wasn't rotational equilibrium.]

(4) Moments can be taken about any point

Consider the rod in the diagram below, subject to the forces F, G & H. If the rod is in equilibrium, then $F + H = G$ (ie there is vertical equilibrium). [In other situations, we can also employ horizontal equilibrium.]



The net moment can be calculated about $A, B \ or \ C$, as follows:

Moments about A [sometimes indicated by: $M(A)$]:

$Ga - H(a + b) = (F + H)a - H(a + b) = Fa - Hb$

Moments about B:

$Fa - Hb$

Moments about C:

$F(a + b + c) - G(b + c) + Hc$

$= F(a + b + c) - (F + H)(b + c) + Hc$

$= Fa - Hb$

More generally, whenever the forces are balanced, taking moments about any point will give the same result.

Then, because the rod is in rotational equilibrium, $Fa - Hb = 0$

(5) Convenient point to take moments about

Some points will be more convenient to take moments about, for the following reasons:

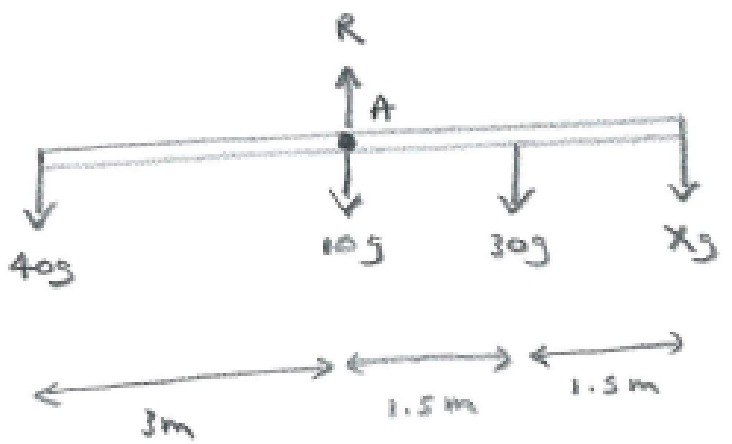
(i) If we are not interested in a particular force, or if it isn't known, then it may be avoided by taking moments about a point through which the force in question acts (eg taking moments about B, in the above example, if G is unknown; note that, in this case, the equation $F + H = G$ would not be used - in order to keep G out of the working).

(ii) Some points involve more complicated equations; eg taking moments about C in the above example. In general, take moments about a point where as many forces as possible act.

(6) Example: Children sitting on a seesaw

The children have masses $40, 30$ & $X\ kg$, and the seesaw (assumed to be uniform) has mass $10kg$.

The problem is to find X, given that the seesaw is in equilibrium.



[R is the reaction force of the supporting structure on the seesaw (and is unknown).]

In this case there is no need to resolve forces vertically (which would give an equation involving R). Instead, we can take moments about A:

Rotational equilibrum ⇒

$40g(3) - 30g(1.5) - Xg(3) = 0$,

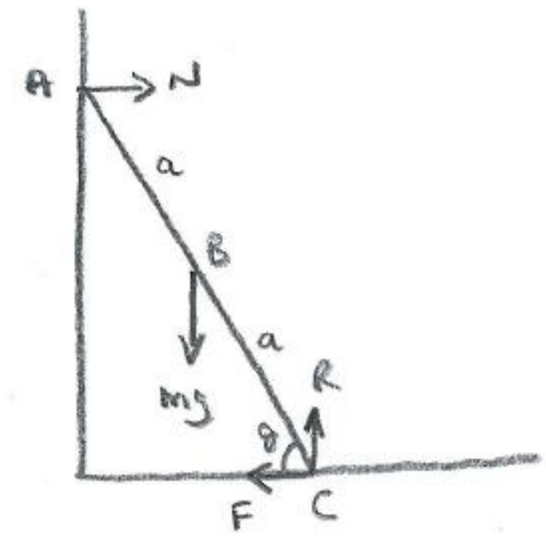
so that $X = \frac{120-45}{3} = 25$

[R can then be found from $R = 40g + 10g + 30g + Xg$, if required.]

Note: As an alternative to equating the net moment to zero, we could say that the total clockwise moment equals the total anti-clockwise moment.

(7) Moments of forces at an angle

Example: Ladder resting against a wall



The ladder is of length $2a$ and mass m, and is assumed to be uniform, so that its centre of mass is at its mid-point. The wall is assumed to be smooth, so that the reaction at the wall, N is perpendicular to the wall. The coefficient of friction between the ladder and the ground is μ. Given that the ladder is resting at an angle θ to the ground, find the minimum possible value of μ in terms of θ.

Approach 1

Resolving vertically, $R = mg$.

If we take moments about A, then one approach is to extend the lines of action of the forces, in order to find the perpendicular (ie shortest) distance between those lines and A.

Thus, the perpendicular distance between mg (extended) and A is $acos\theta$; between R (extended) and A: $2acos\theta$, and between F (extended) and A: $2asin\theta$.

Then rotational equilibrium ⇒ the net moment about A is zero,

so that $-mg(acos\theta) + R(2acos\theta) - F(2asin\theta) = 0$

Also, in the limiting case, where the ladder is about to slip,

$F = \mu R = \mu mg$

Thus $-cos\theta + 2cos\theta - 2\mu sin\theta = 0$,

so that $\mu = \frac{cos\theta}{2sin\theta} = \frac{1}{2}cot\theta$

(Note that, for larger θ, a smaller value of μ will be sufficient to keep the ladder in place.)

Approach 2

An alternative way of finding the moments of the forces is to resolve each force, at a suitable point on its line of action, in two convenient perpendicular directions.

Thus, mg can be resolved at B into components along and perpendicular to the ladder. The component along the ladder then has no moment about A, whilst the component perpendicular to the ladder ($mgcos\theta$) has moment $-(mgcos\theta)a$.

Similarly, R has moment $(Rcos\theta)(2a)$, whilst F has moment

$-(Fsin\theta)(2a)$.

Thus each of the moment terms is the same as before.

Note: In the case of the mg force, B is the best place to resolve the force (as one of the components has zero moment about A). However, it can be shown that the same total moment would be obtained if the force were resolved at some other point on its line of action.

(8) Alternative approaches

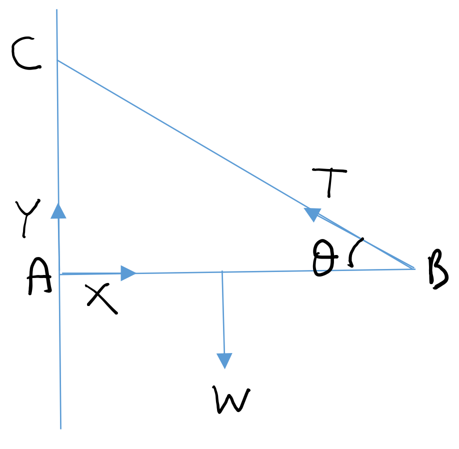
Once forces have been resolved in two perpendicular directions and moments taken about a particular point, so that 3 equations have been created, it isn't possible to obtain an independent 4th equation by taking moments about another point; ie it will just duplicate information already obtained.

However, it is possible to take moments about 2 points and resolve forces in just one direction - provided that this direction isn't perpendicular to the line joining the 2 points.

Alternatively, it is possible to take moments about 3 points (and do no resolving of forces) - provided that the 3 points don't lie on a straight line.

As it is usually simpler to resolve forces, rather than take moments, these alternative methods are not normally used. They could be used as a check though.

Example: Horizontal rod AB, of length a, attached to wall at A, and to rope BC at B



Method 1

Resolving horizontally: $X = Tcos\theta$

Resolving vertically: $Y = W - Tsin\theta$

Taking moments about A: $Tsin\theta.a = W(\frac{a}{2})$

So $Y = W - \frac{W}{2} = \frac{W}{2}$,

$T = \frac{Wcosec\theta}{2}$

and $X = \frac{Wcot\theta}{2}$

Method 2

Resolving horizontally: $X = Tcos\theta$

Taking moments about A: $Tsin\theta.a = W(\frac{a}{2})$

Taking moments about B: $Ya = W\left(\frac{a}{2}\right)$

[Note: The horizontal direction is not perpendicular to the line joining A and B.]

So $Y = \frac{W}{2}$, $T = \frac{Wcosec\theta}{2}$ and $X = \frac{Wcot\theta}{2}$

Method 3

Taking moments about A: $Tsin\theta . a = W(\frac{a}{2})$

Taking moments about B: $Ya = W\left(\frac{a}{2}\right)$

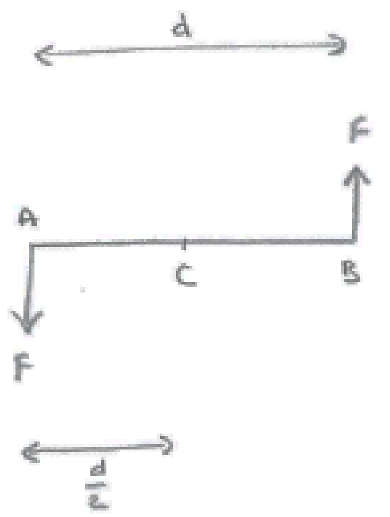
Taking moments about C: $X. atan\theta = W(\frac{a}{2})$

So $T = \frac{Wcosec\theta}{2}$, $Y = \frac{W}{2}$ and $X = \frac{Wcot\theta}{2}$

(9) Couples

The term 'couple' is used to describe a pair of equal but opposite forces, applied to an object, which don't have the same line of action, so that there is a turning effect. (It is sometimes also used in the more general situation where there are more than two forces, which have a resultant of zero but a net non-zero moment.)

As before, the fact that the forces are balanced means that it doesn't matter which point we take moments about. Thus, referring to the diagram below, we could take moments about A, for example, to give a net moment of Fd.



(10) Hinged Joints

Suppose that a rod is attached to a wall by a hinged joint (ie so that the angle can be varied). The hinge will often be described as 'smooth' or 'free'. This means that it offers no resistance to being turned; ie there is no moment within the hinge countering any external forces. (Were the hinge not to be smooth then the resistance to turning within the hinge could be thought of as due to the moment of a frictional force acting at a short distance from the centre of the hinge.)

(11) Reaction forces at a surface

If a rod, say, is attached to a surface, then there will be a reaction force on the rod, at a particular angle. In practice, it is usually convenient to resolve this reaction force into two perpendicular components: along and perpendicular to the surface. Were the rod to be resting on the surface (say, if it were a ladder placed against a wall), then the component along the surface would be the frictional force (and this is taken to be zero if the wall is smooth).

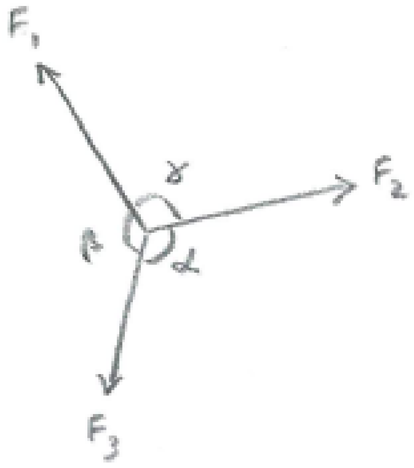
(12) Alternative equilibrium methods

(i) In the case of 3 forces (with no forces being parallel), the lines of action must meet at a point (be 'concurrent').

(ii) If the forces are represented by vector arrows, then they will form a vector polygon.

(iii) Lami's theorem (similar to the Sine rule)

$$\frac{F_1}{sin\alpha} = \frac{F_2}{sin\beta} = \frac{F_3}{sin\gamma}$$



F_1
F_2
F_3
γ
β
α