Matrices – Q58: Transformations (9/3/24)

Suppose that the matrix $M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ represents a reflection in the line $y = mx$.

(i) By considering the image of a particular point under M, explain why $a^2 + b^2 = 1$.

(ii) Given that a necessary and sufficient condition for a transformation to have a line of invariant points is that $tr(M) = |M| + 1$ (where $tr(M)$ (the trace of M) $= a + d$) and by considering the image of a point on the line $y = mx$ (or otherwise), show that $M = \begin{pmatrix} \frac{1-m^2}{1+m^2} & \frac{2m}{1+m^2} \\ \frac{2m}{1+m^2} & \frac{m^2-1}{1+m^2} \end{pmatrix}$

Solution

(i) $\binom{a}{b}$ is the image of $\binom{1}{0}$, and the distance of $\binom{a}{b}$ from the Origin

$(a^2 + b^2)$ will equal the distance of $\binom{1}{0}$ from the Origin (1);

thus $a^2 + b^2 = 1$

(ii) When any shape is reflected in the line $y = mx$, the area will be unchanged, but the order of the vertices will be reversed (so that clockwise becomes anti-clockwise). Hence $|M| = -1$.

Also, from the given result,

$a + d = |M| + 1 = -1 + 1 = 0$

So we can write $M = \begin{pmatrix} a & \frac{1-a^2}{b} \\ b & -a \end{pmatrix}$, where $a^2 + b^2 = 1$.

As $\binom{1}{m}$ lies on the line of reflection,

$\begin{pmatrix} a & \frac{1-a^2}{b} \\ b & -a \end{pmatrix}\binom{1}{m} = \binom{1}{m}$,

and so $a + \left(\frac{1-a^2}{b}\right)m = 1$, and $b - am = m$

Thus $b = m(a+1)$ [and $a + (1-a) = 1$]

Then $M = \begin{pmatrix} a & \frac{1-a^2}{m(a+1)} \\ m(a+1) & -a \end{pmatrix}$

and $a^2 + m^2(a+1)^2 = 1$,

so that $m^2 = \frac{1-a^2}{(1+a)^2} = \frac{1-a}{1+a}$;

$m^2 + am^2 = 1 - a;$

$a(m^2 + 1) = 1 - m^2;$

$a = \frac{1-m^2}{1+m^2}$

Then $b = m(a+1) = \frac{m(1-m^2)+m(1+m^2)}{1+m^2} = \frac{2m}{1+m^2}$

And $c = \frac{1-a^2}{m(a+1)} = \frac{(1-a)}{m} = \frac{1}{m}.\frac{(1+m^2)-(1-m^2)}{1+m^2} = \frac{2m}{1+m^2}$

So $M = \begin{pmatrix} \frac{1-m^2}{1+m^2} & \frac{2m}{1+m^2} \\ \frac{2m}{1+m^2} & \frac{m^2-1}{1+m^2} \end{pmatrix}$, as required.