Matrices – Q57: Transformations (9/3/24)

Consider the transformation represented by the matrix

$M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$.

(i) Show that the transformation has at least one invariant line

if and only if $[tr(M)]^2 \geq 4|M|$

(where $tr(M)$ (the trace of M) $= a + d$)

(ii) Show that there will be a family of invariant lines if and only if

$tr(M) = |M| + 1$

Solution

(i) Suppose that there is an invariant line $y = mx + k$.

Then $\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} x \\ mx+k \end{pmatrix} = \begin{pmatrix} ax+cmx+ck \\ bx+dmx+dk \end{pmatrix}$

and $bx + dmx + dk = m(ax + cmx + ck) + k$

This must apply for all x, so:

equating coefficients of x: $b + dm = ma + cm^2$,

so that $cm^2 + m(a-d) - b = 0$ (1)

And equating constant terms gives $dk = mck + k$,

so that $k(d - mc - 1) = 0$ (2)

Then, in order for there to be an m that satisfies (1),

the discriminant of (1) must be non-negative (and vice-versa);

ie $(a-d)^2 + 4cb \geq 0$

$\Leftrightarrow (a+d)^2 - 4ad + 4cb \geq 0$

$\Leftrightarrow [tr(M)]^2 \geq 4|M|$, as required.

(ii) From (2), if the condition in (i) is satisfied, there will only be one invariant line (with $k = 0$) unless $d - mc - 1 = 0$;

ie $m = \frac{d-1}{c}$

Then, from (1): $cm^2 + m(a-d) - b = 0$,

so that $(d-1)^2 + (d-1)(a-d) - bc = 0$

or $d^2 - 2d + 1 + da - d^2 - a + d - bc = 0$

or $-d + 1 + ad - a - bc = 0$

or $tr(M) = |M| + 1$, as required.

Then $[tr(M)]^2 - 4|M| = [|M| + 1]^2 - 4|M|$

$= [|M| - 1]^2 \geq 0$,

so that the condition in (i) is satisfied.

This argument is reversible, and so there will be a family of invariant lines if and only if $tr(M) = |M| + 1$.