Matrices – Simultaneous Equations

Q55 [Problem/M] (3/11/23)

Given that $a + 2b = 16$ and $b - c = 2$, use a matrix argument to determine that $a + b + c = 14$

Solution

[Note that $(a + 2b) - (b - c) = a + b + c$,

so that $a + b + c = 16 - 2 = 14$]

We need to solve the simultaneous equations

$a + 2b = 16$

$b - c = 2$

$a + b + c = k$,

and establish k in the process (assuming that a unique value of k exists).

These equations can be written as $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 16 \\ 2 \\ k \end{pmatrix}$,

and we see that $\begin{vmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{vmatrix} = 1(2) - 2(1) = 0$ (expanding by the 1st row).

Clearly there are an infinite number of solutions to the first two equations (assuming that there are no restrictions on a, b & c), so that the 3 equations must have a solution for at least one value of k.

For this to be the case, we require $\begin{vmatrix} 1 & 2 & 16 \\ 0 & 1 & 2 \\ 1 & 1 & k \end{vmatrix} = 0$

[This 3d result corresponds to the requirement for the following 2d equations to be consistent when $\begin{vmatrix} a & c \\ b & d \end{vmatrix} = 0$:

$ax + cy = e$

$bx + dy = f$

As $\frac{a}{b} = \frac{c}{d}$, we require $\frac{e}{f} = \frac{a}{b}$, or $af = be$; ie $\begin{vmatrix} a & e \\ b & f \end{vmatrix} = 0$]

Expanding by the 3rd row gives:

$1(-12) - 1(2) + k(1) = 0$, so that $k = 14$

[Note: As we have established that there is a solution to the equations (with $k = 14$), the fact that $\begin{vmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{vmatrix} = 0$ means that there will be an infinite number of solutions. In fact, any value can be chosen for a (or b or c), and the total $a + b + c$ always equals 14.]