Matrices – Q51: Eigenvectors [Practice/M] (22/7/21)

Consider the transformation represented by the matrix $\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$

(i) Write down the eigenvalues, without forming the characteristic equation.

(ii) Find the invariant lines passing through the Origin.

(iii) Show that all points in the plane are transformed to points on a specific line (to be found).

(iv) Find the line whose points are transformed to $(1,3)$.

(v) State the line whose points are transformed to the Origin.

Solution

(i) $\lambda_1 + \lambda_2 = 1 + 6 = 7$ & $\lambda_1\lambda_2 = \begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix} = 0$

Hence $\lambda_1 = 0$ & $\lambda_2 = 7$

(ii) **For $\boldsymbol{\lambda = 0}$:**

$$\begin{pmatrix} 1-0 & 2 \\ 3 & 6-0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

so that $x + 2y = 0$, giving an eigenvector of $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$

For $\boldsymbol{\lambda = 7}$:

$$\begin{pmatrix} 1-7 & 2 \\ 3 & 6-7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

so that $-6x + 2y = 0$, giving an eigenvector of $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$

Thus the invariant lines through the Origin are

$y = -\frac{1}{2}x$ & $y = 3x$

(iii) [As $\begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix} = 0$, the area scale factor of the transformation is 0, which means that all shapes collapse to points on a particular line]

$$\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$$

$\Rightarrow p + 2q = u$

and $3p + 6q = v$

so that $v = 3u$; equation of line is $y = 3x$ (ie one of the eigenvectors, and invariant lines)

(iv) $\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \Rightarrow p + 2q = 1$

$\Rightarrow q = \frac{1}{2}(1 - p)$

ie equation is $y = \frac{1}{2} - \frac{x}{2}$

(v) The other eigenvector, $y = -\frac{1}{2}x$ has an eigenvalue of 0, and so all its points are transformed to the Origin [$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow 0\begin{pmatrix} x \\ y \end{pmatrix}$]