Matrices – Q45: Invariant Points & Lines [H] (9/3/24)

For the transformation matrix $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$, where a, b, c & d are positive, find a relationship between the trace $a + d$ and the determinant that must hold in order for the transformation to have an invariant line that doesn't pass through the Origin.

Solution

Suppose that $\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} x \\ mx+k \end{pmatrix} = \begin{pmatrix} x' \\ mx'+k \end{pmatrix}$ for all x,

where $k \neq 0$

Then $ax + cmx + ck = x'$ & $bx + dmx + dk = mx' + k$

and $(a + cm)x + ck = x'$ & $(b + dm)x + (d - 1)k = mx'$

Multiplying the 1st equation by m and equating the two expressions for mx' gives:

$m(a + cm)x + mck = (b + dm)x + (d - 1)k$

As this is to hold for all x, we can equate the coefficients of x , to give:

$m(a + cm) = b + dm$ & $mck = (d - 1)k \quad (1)$

Thus, as k & $c \neq 0$,

$cm^2 + (a - d)m - b = 0$ & $m = \frac{d-1}{c}$,

and hence $c\left(\frac{d-1}{c}\right)^2 + (a - d)\left(\frac{d-1}{c}\right) - b = 0$,

so that $(d - 1)^2 + (a - d)(d - 1) - bc = 0$

and $(d - 1)(d - 1 + a - d) - bc = 0$,

giving $(d - 1)(a - 1) - bc = 0$

and hence $ad - bc - (a + d) + 1 = 0$

or $trace = det + 1$