Matrices – Q38: Simultaneous Eq'ns [Problem/M]
(4/6/21)

Consider the planes with the following equations:

$$\begin{aligned} ax - y + z &= 1 \\ 2y - z &= b \\ 4x + 3y - 2z &= 2 \end{aligned}$$

(i) Find conditions on a and b for:
(a) the 3 planes to meet at a single point
(b) the 3 planes to meet in a line
(c) no point of intersection of the 3 planes

(ii) Show that in case (c) the line of intersection of the 1st two planes is parallel to the 3rd plane.

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Solution

(a) $\Delta = \begin{vmatrix} a & -1 & 1 \\ 0 & 2 & -1 \\ 4 & 3 & -2 \end{vmatrix} =$ (expanding by the 1st column)

$a(-1) + 4(-1)$

The 3 planes will meet at a single point when $\Delta \neq 0$; ie when $a \neq -4$

(b) The 3 planes will meet in a line (as a sheaf of planes) when $\Delta = 0$ (ie $a = -4$) and the equations are consistent.

Method 1

$$\begin{aligned} -4x - y + z &= 1 \quad (1) \\ 2y - z &= b \quad (2) \\ 4x + 3y - 2z &= 2 \quad (3) \end{aligned}$$

(1) + (3) gives $2y - z = 3$

This is consistent with (2) when $b = 3$

Method 2

Replacing eg the 3rd column of the determinant with $\begin{pmatrix} 1 \\ b \\ 2 \end{pmatrix}$,

the equations will be consistent when $\begin{vmatrix} -4 & -1 & 1 \\ 0 & 2 & b \\ 4 & 3 & 2 \end{vmatrix} = 0$

(this is a result connected with Cramer's method for solving simultaneous equations)
Expanding about the 2nd row,

$$\begin{vmatrix} -4 & -1 & 1 \\ 0 & 2 & b \\ 4 & 3 & 2 \end{vmatrix} = 0 \Rightarrow 2(-12) - b(-8) = 0 \Rightarrow b = 3$$

Method 3

We can attempt to find a common line:

Let (eg) $x = \lambda$, so that

$-y + z = 1 + 4\lambda$ (1)
$2y - z = b$ (2)
$3y - 2z = 2 - 4\lambda$ (3)

Then (1) + (2) gives $y = 1 + b + 4\lambda$

and (2) + 2(1) gives $z = b + 2 + 8\lambda$

Substituting into (3): $3(1 + b + 4\lambda) - 2(b + 2 + 8\lambda) = 2 - 4\lambda$

$\Rightarrow -3 + b = 0 \Rightarrow b = 3$

(c) There will be no point of intersection of the 3 planes (which will then form a triangular prism) when $a = -4$ and $b \neq 3$

(ii) To find the intersection of the first two planes:

$$-4x - y + z = 1$$
$$2y - z = b$$

Let eg $x = \lambda$, so that:

$$-y + z = 1 + 4\lambda \quad (1)$$
$$2y - z = b \quad (2)$$

(1) + (2) gives $y = 1 + 4\lambda + b$

And 2(1) + (2) gives $z = 2 + 8\lambda + b$

Hence the line of intersection of the first two planes is:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1+b \\ 2+b \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 4 \\ 8 \end{pmatrix}$$

Then $\begin{pmatrix} 1 \\ 4 \\ 8 \end{pmatrix} . \begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix} = 4 + 12 - 16 = 0$,

so that this line is perpendicular to the normal vector to the 3rd plane, and hence parallel to the plane.