Matrices – Q37: Simultaneous Eq’ns [Practice/E] (3/6/21)

Show that the following three planes meet in a line, giving the equation of that line in cartesian form.

$x - y + 3z = 4$
$4x + 5y - 2z = 8$
$x + 17y - 25z = -12$

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Solution

First of all, none of the lines are parallel to each other.

Then $\begin{vmatrix} 1 & -1 & 3 \\ 4 & 5 & -2 \\ 1 & 17 & -25 \end{vmatrix} = 1(-91) - (-1)(-98) + 3(63) = 0$

[as expected for this sort of question]

So the planes will either be configured as a sheaf (if they have a line of intersection) or as a triangular prism (if not).

[In some cases it may be possible to spot that one equation is a combination of the other two, showing that the equations are consistent, and that they meet in a line.]

$x - y + 3z = 4 \quad (1)$
$4x + 5y - 2z = 8 \ (2)$
$x + 17y - 25z = -12 (3)$

Substituting for x (say), from (1) into (2) gives:

$4(4 + y - 3z) + 5y - 2z = 8$, so that $9y - 14z = -8$

Substituting into (3) gives:

$(4 + y - 3z) + 17y - 25z = -12$, so that $18y - 28z = -16$,

which is the same equation, and hence the planes meet as a sheaf.

To find the line of intersection, let $x = \lambda$ (say).

Then, from (1), $-y + 3z = 4 - \lambda$ (3)

and from (2), $5y - 2z = 8 - 4\lambda$ (4)

Then $5(3) + (4) \Rightarrow 13z = 28 - 9\lambda$

and $2(3) + 3(4) \Rightarrow 13y = 32 - 14\lambda$,

so that the equation of the line is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{13}\begin{pmatrix} 13\lambda \\ 32 - 14\lambda \\ 28 - 9\lambda \end{pmatrix}$

or $\frac{x}{13} = \frac{y - \frac{32}{13}}{-14} = \frac{z - \frac{28}{13}}{-9}$

[As a check, points on the line where $\lambda = 0$ and 1 could be substituted into the equations of the planes.

Also, it can be shown that the determinant formed by replacing (any) one of the columns of the matrix by the right-hand values will be zero when the equations are consistent. (Consider the 2×2 case to see why this is likely to be true.)

Thus $\begin{vmatrix} 1 & -1 & 4 \\ 4 & 5 & 8 \\ 1 & 17 & -12 \end{vmatrix} = 1(-196) - (-1)(-56) + 4(63) = 0$, for example.]