Matrices – Q35: Simultaneous Eq'ns [Practice/E] (3/6/21)

(i) Three planes are represented by the following equations:

$x - y + z = 1$

$2x + ky + 2z = 3$

$x + 3y + 3z = 5$

For what value of k do the planes not meet at a single point? For this value of k how are the planes configured?

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Solution

(i) $\begin{pmatrix} 1 & -1 & 1 \\ 2 & k & 2 \\ 1 & 3 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$

$\begin{vmatrix} 1 & -1 & 1 \\ 2 & k & 2 \\ 1 & 3 & 3 \end{vmatrix} = (3k - 6) + (6 - 2) + (6 - k)$ [expanding by the 1st row]

$= 2k + 4$

The equations don't have a unique solution when $2k + 4 = 0$; ie $k = -2$

In that case, the equations are:

$x - y + z = 1$

$2x - 2y + 2z = 3$

$x + 3y + 3z = 5$

As the direction vectors of the first two planes are $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix}$, which are equivalent, and the constant terms on the RHS are not in the same ratio as the LHS terms, these planes are parallel, and the 3rd plane cuts both of the other planes (not being parallel to either of them).

(ii) To solve $\begin{pmatrix} 1 & -1 & 1 \\ 2 & 2 & 2 \\ 1 & 3 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$: det $= 2k + 4 = 8$

and so $\begin{pmatrix} 1 & -1 & 1 \\ 2 & 2 & 2 \\ 1 & 3 & 3 \end{pmatrix}^{-1} = \frac{1}{8}\begin{pmatrix} 0 & -4 & 4 \\ 6 & 2 & -4 \\ -4 & 0 & 4 \end{pmatrix}^{T} =$
$\frac{1}{8}\begin{pmatrix} 0 & 6 & -4 \\ -4 & 2 & 0 \\ 4 & -4 & 4 \end{pmatrix}$

[eg $6 = -\big((-1) \times 3 - 3 \times 1\big)$; $2 = 1 \times 3 - 1 \times 1$;
$-4 = -\big(1 \times 3 - 1 \times (-1)\big)$]

So $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{8}\begin{pmatrix} 0 & 6 & -4 \\ -4 & 2 & 0 \\ 4 & -4 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} = \frac{1}{8}\begin{pmatrix} -2 \\ 2 \\ 12 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} -1 \\ 1 \\ 6 \end{pmatrix}$

ie $x = \frac{-1}{4}$, $y = \frac{1}{4}$, $z = \frac{3}{2}$