

Matrices – Q30: Invariant Points & Lines [M] (9/3/24)

$M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ represents a transformation.

Under what conditions will there be a line of invariant points passing through the Origin?

[It can in fact be shown that any line of invariant points will pass through the Origin.]

Solution

Suppose that there is a line of invariant points $y = mx$,

so that $\begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} x \\ mx \end{pmatrix} = \begin{pmatrix} x \\ mx \end{pmatrix}$ for all x

ie $\begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} x \\ mx \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ mx \end{pmatrix}$

or $\begin{pmatrix} a-1 & c \\ b & d-1 \end{pmatrix} \begin{pmatrix} x \\ mx \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

For there to be a solution other than $x = 0, y = 0$,

$$\begin{vmatrix} a-1 & c \\ b & d-1 \end{vmatrix} = 0$$

$$\Rightarrow (a-1)(d-1) - bc = 0$$

$$\Rightarrow 1 - (a+d) + ad - bc = 0$$

$$\Rightarrow \text{tr}M = |M| + 1$$