Matrices – Q21: General [Problem/H](2/6/21)

Suppose that the following pair of equations enables (x', t') to be determined from (x, t):

$$x' = \gamma(x - vt) \;\&\; t' = \gamma(t - \frac{xv}{c^2}) \quad \text{(A)}$$

and that it is also true that

$$x = \gamma(x' + vt') \;\&\; t = \gamma(t' + \frac{x'v}{c^2}) \quad \text{(B)}$$

[These are the transformation equations in Special Relativity between two frames of reference that are moving with a relative speed of v. Starting with (A), (B) is obtained by reversing the roles of the two frames (so that the speed is reversed as well).]

Use matrix multiplication to find an expression for γ in terms of v & c.

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Solution

$$x' = \gamma(x - vt) \;\&\; t' = \gamma(t - \frac{xv}{c^2})$$

$$\Rightarrow \begin{pmatrix} x' \\ t' \end{pmatrix} = \gamma \begin{pmatrix} 1 & -v \\ -\frac{v}{c^2} & 1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix}$$

and $x = \gamma(x' + vt') \;\&\; t = \gamma(t' + \frac{x'v}{c^2})$

$$\Rightarrow \begin{pmatrix} x \\ t \end{pmatrix} = \gamma \begin{pmatrix} 1 & v \\ \frac{v}{c^2} & 1 \end{pmatrix} \begin{pmatrix} x' \\ t' \end{pmatrix}$$

Hence $\begin{pmatrix} x' \\ t' \end{pmatrix} = \gamma \begin{pmatrix} 1 & -v \\ -\frac{v}{c^2} & 1 \end{pmatrix} \gamma \begin{pmatrix} 1 & v \\ \frac{v}{c^2} & 1 \end{pmatrix} \begin{pmatrix} x' \\ t' \end{pmatrix}$

and so $\gamma \begin{pmatrix} 1 & -v \\ -\frac{v}{c^2} & 1 \end{pmatrix} \gamma \begin{pmatrix} 1 & v \\ \frac{v}{c^2} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

giving $\gamma^2 \begin{pmatrix} 1-\frac{v^2}{c^2} & 0 \\ 0 & 1-\frac{v^2}{c^2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

and hence $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$

[This is the Lorentz factor.]