Matrices – Q15: Eigenvectors [Problem/H](2/6/21)

The populations of sparrows (x) and sparrowhawks (y) in a particular area satisfy the following differential equations:

$$\frac{dx}{dt} = 0.1x - 2y \quad \text{and} \quad \frac{dy}{dt} = 0.1x + y$$

(where time is measured in years),

and initially there are 50 sparrows and 4 sparrowhawks.

The equations can be written as $\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} 0.1 & -2 \\ 0.1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ (*)

(i) Express $\begin{pmatrix} 0.1 & -2 \\ 0.1 & 1 \end{pmatrix}$ in the form PDP^{-1}, where D is a diagonal matrix.

(ii) Show that (*) can be rewritten as as $\begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix} = D \begin{pmatrix} u \\ v \end{pmatrix}$

(iii) Show that $u = Ae^{0.6t}$ and $v = Be^{0.5t}$, where A and B are arbitrary constants, and hence solve the original differential equations.

(iv) What happens to the two populations?

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Solution

(i) The columns of P will be the eigenvectors of the matrix, and the non-zero elements of D will be the eigenvalues.

The eigenvalues satisfy $\begin{pmatrix} 0.1 & -2 \\ 0.1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$

and the characteristic equation is $(0.1 - \lambda)(1 - \lambda) - 0.1(-2) = 0$,

so that $\lambda^2 - 1.1\lambda + 0.3 = 0$

$\Rightarrow (\lambda - 0.6)(\lambda - 0.5) = 0$

$\Rightarrow \lambda = 0.6 \ \ or \ \ 0.5$

When $\lambda = 0.6$,

$\begin{pmatrix} 0.1 & -2 \\ 0.1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0.6\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow 0.1x - 2y = 0.6x \Rightarrow y = -0.25x$

When $\lambda = 0.5$,

$\begin{pmatrix} 0.1 & -2 \\ 0.1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0.5\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow 0.1x - 2y = 0.5x \Rightarrow y = -0.2x$

So $P = \begin{pmatrix} 4 & 5 \\ -1 & -1 \end{pmatrix}$ and $D = \begin{pmatrix} 0.6 & 0 \\ 0 & 0.5 \end{pmatrix}$

Then $P^{-1} = \frac{1}{1}\begin{pmatrix} -1 & -5 \\ 1 & 4 \end{pmatrix}$

and $\begin{pmatrix} 0.1 & -2 \\ 0.1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 5 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 0.6 & 0 \\ 0 & 0.5 \end{pmatrix}\begin{pmatrix} -1 & -5 \\ 1 & 4 \end{pmatrix}$

(ii) $\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = PDP^{-1}\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow P^{-1}\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = DP^{-1}\begin{pmatrix} x \\ y \end{pmatrix}$ (**)

Let $\begin{pmatrix} u \\ v \end{pmatrix} = P^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & -5 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x - 5y \\ x + 4y \end{pmatrix}$

Then $\begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix} = \begin{pmatrix} -\frac{dx}{dt} - 5\frac{dy}{dt} \\ \frac{dx}{dt} + 4\frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} -1 & -5 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = P^{-1}\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix}$

so that $\begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix} = D\begin{pmatrix} u \\ v \end{pmatrix}$, from (**)

(iii) $\begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix} = \begin{pmatrix} 0.6 & 0 \\ 0 & 0.5 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \Rightarrow \frac{du}{dt} = 0.6u \ \ \& \ \frac{dv}{dt} = 0.5v$

$\Rightarrow \int \frac{1}{u} du = 0.6 \int dt \Rightarrow ln|u| = 0.6t + C$

$\Rightarrow u = Ae^{0.6t}$, and similarly $v = Be^{0.5t}$

$\begin{pmatrix} u \\ v \end{pmatrix} = P^{-1} \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = P \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 4 & 5 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} Ae^{0.6t} \\ Be^{0.5t} \end{pmatrix}$

so that $x = 4Ae^{0.6t} + 5Be^{0.5t}$ and $y = -Ae^{0.6t} - Be^{0.5t}$

Applying the initial conditions,

$50 = 4A + 5B$ and $4 = -A - B$

$\Rightarrow 50 = 4A + 5(-A - 4) \Rightarrow 70 = -A$ and $B = 66$

So $x = 330e^{0.5t} - 280e^{0.6t}$ and $y = 70e^{0.6t} - 66e^{0.5t}$

(iv) The sparrows become extinct when $x = 0$

$\Rightarrow 330e^{0.5t} = 280e^{0.6t} \Rightarrow \frac{33}{28} = e^{0.1t}$

$\Rightarrow t = 10ln\left(\frac{33}{28}\right) = 1.643$ years; ie by the time 20 months have elapsed

($y = 0 \Rightarrow 70e^{0.6t} = 66e^{0.5t} \Rightarrow \frac{66}{70} = e^{0.1t} \Rightarrow t = 10ln\left(\frac{66}{70}\right) < 0$, which can be rejected)