Matrices Overview (9/3/24; 17 pages)

General

Q16 [Problem/E]

Find k such that $\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$ and $\begin{pmatrix} 5 & 9 \\ 6 & k \end{pmatrix}$ commute.

Q21 [Problem/H]

Suppose that the following pair of equations enables (x', t') to be determined from (x, t):

$x' = \gamma(x - vt)$ & $t' = \gamma(t - \frac{xv}{c^2})$ (A)

and that it is also true that

$x = \gamma(x' + vt')$ & $t = \gamma(t' + \frac{x'v}{c^2})$ (B)

[These are the transformation equations in Special Relativity between two frames of reference that are moving with a relative speed of v. Starting with (A), (B) is obtained by reversing the roles of the two frames (so that the speed is reversed as well).]

Use matrix multiplication to find an expression for γ in terms of v & c.

Q25 [Problem/H]

Find the condition(s) for two 2×2 matrices to commute.

Determinants

Q18 [Problem/M]

If $M = \begin{pmatrix} \lambda & k \\ 1 & \lambda - k \end{pmatrix}$, where λ & k are real numbers, what is the range of possible values of k, in order that $|M| > 0$ for all values of λ?

Q23 [Practice/M]

Factorise the determinant $\begin{vmatrix} x^2 - x & y^2 - y & z^2 - z \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$

Q24 [Practice/M]

Write the determinant $\begin{vmatrix} \mathbf{1} & \boldsymbol{x^2} & \boldsymbol{x^4} \\ \mathbf{1} & \boldsymbol{y^2} & \boldsymbol{y^4} \\ \mathbf{1} & \boldsymbol{z^2} & \boldsymbol{z^4} \end{vmatrix}$ as a product of linear factors.

Q26 [Problem/H]

Given that a 3×3 determinant can always be reduced to triangular form (in the same way as simultaneous equations), to produce a multiple of $\begin{vmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{vmatrix}$, show that it can be further reduced to a multiple of the Identity matrix. [Obviously this is an academic exercise, as the determinant can be evaluated as soon as triangular form has been reached.]

Eigenvectors

Q4 [Practice/M]

Find $\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}^3$, using eigenvectors.

Q5 [Problem/M]

If matrices M & N (both square, of the same order) share an eigenvector, what can be said about the eigenvectors and eigenvalues of $MN\ and\ NM$?

Q6 [Problem/H]

Given that the eigenvalues of $\begin{pmatrix} 3 & -1 & 1 \\ -1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}$ are 4, 4 and 1,

establish the geometrical significance of the eigenvectors.

Q7 [Problem/H]

(i) Show that the eigenvalues of the matrix $\begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & 2 \end{pmatrix}$ are 1 (repeated) and 2 (for example, by using row or column operations), and investigate the geometrical significance of the eigenvectors.

(ii) Construct another matrix with the same eigenvalues, and hence establish that the geometrical result in (i) does not hold in general.

Q8 [Problem/H]

For a 3×3 matrix M, show that

(i) the product of the eigenvalues of M equals $\det M$

(ii) the sum of the eigenvalues equals the sum of the elements on the leading diagonal of M (from top left to bottom right; this sum is called the trace of M, or trM)

Q9 [Problem/H]

(i) If $\underline{s}_1, \underline{s}_2$ & $\underline{s}_3$ are eigenvectors corresponding to distinct eigenvalues λ_1, λ_2 & λ_3 of a 3×3 matrix M, prove that $\underline{s}_1, \underline{s}_2$ & $\underline{s}_3$ cannot be coplanar.

(ii) Deduce that a 3×3 matrix with distinct eigenvalues can always be diagonalised.

Q10 [Problem/H]

Matrices A & B are said to be 'similar' if $B = PAP^{-1}$ for some matrix P (A need not be diagonal).

Prove that similar matrices have the same characteristic equation, and hence the same eigenvalues.

Q11 [Problem/H]

Symmetric matrices are always diagonalisable. Prove that this is the case for 2×2 symmetric matrices.

Q12 [Problem/H]

Prove that if M is orthogonally diagonalisable, then M is symmetric.

Q13 [Problem/H]

Show that 2×2 matrices representing rotations are not diagonalisable.

Q14 [Problem/M]

For the matrix $M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ with eigenvalues λ_1 & λ_2, prove that $\lambda_1 + \lambda_2 = a + d$, and also that $\lambda_1\lambda_2 = |M|$

[this can be extended to 3×3 matrices]

Q15 [Problem/H]

The populations of sparrows ($\boldsymbol{x}$) and sparrowhawks ($\boldsymbol{y}$) in a particular area satisfy the following differential equations:

$\boldsymbol{\frac{dx}{dt} = 0.1x - 2y}$ and $\boldsymbol{\frac{dy}{dt} = 0.1x + y}$

(where time is measured in years),

and initially there are 50 sparrows and 4 sparrowhawks.

The equations can be written as $\boldsymbol{\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} 0.1 & -2 \\ 0.1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}}$ (*)

(i) Express $\begin{pmatrix} \mathbf{0.1} & \mathbf{-2} \\ \mathbf{0.1} & \mathbf{1} \end{pmatrix}$ in the form $\boldsymbol{PDP^{-1}}$, where $\boldsymbol{D}$ is a diagonal matrix.

(ii) Show that (*) can be rewritten as as $\begin{pmatrix} \frac{\boldsymbol{du}}{\boldsymbol{dt}} \\ \frac{\boldsymbol{dv}}{\boldsymbol{dt}} \end{pmatrix} = \boldsymbol{D}\begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{pmatrix}$

(iii) Show that $\boldsymbol{u} = \boldsymbol{A}\boldsymbol{e}^{\mathbf{0.6}\boldsymbol{t}}$ and $\boldsymbol{v} = \boldsymbol{B}\boldsymbol{e}^{\mathbf{0.5}\boldsymbol{t}}$, where $\boldsymbol{A}$ **and** $\boldsymbol{B}$ are arbitrary constants, and hence solve the original differential equations.

(iv) What happens to the two populations?

Q47 [Practice/M]

For a reflection in the line $y = x$:

(i) Find the transformation matrix.

(ii) Use eigenvectors to find the invariant lines through the Origin.

(iii) By drawing a diagram, are there any invariant lines that don't pass through the Origin?

(iv) Are there any lines of invariant points?

Q48 [Practice/M]

Consider the transformation represented by $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

(i) What type of transformation is this?

(ii) Use eigenvectors to find the invariant lines through the Origin.

(iii) What can be said about the line $y = 0$?

(iv) What can be said about the line $y = c$? , where $c \neq 0$

Q51 [Practice/M]

Consider the transformation represented by the matrix $\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$

(i) Write down the eigenvalues, without forming the characteristic equation.

(ii) Find the invariant lines passing through the Origin.

(iii) Show that all points in the plane are transformed to points on a specific line (to be found).

(iv) Find the line whose points are transformed to $(1,3)$.

(v) State the line whose points are transformed to the Origin.

Invariant Points & Lines

Q27 [Practice/E]

Find the value of k for which the transformation $\begin{pmatrix} 2 & 4 \\ 3 & k \end{pmatrix}$ has a line of invariant points, and find this line.

Q28 [Practice/M]

(i) Use a matrix method to find the invariant lines for a reflection in the y-axis.

(ii) Investigate the invariant lines for a reflection in the x-axis.

Q29 [Problem/M]

$M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ represents a transformation.

(i) Under what conditions will $x = 0$ be an invariant line?

(ii) Under what conditions will there be an invariant line of the form $x = \lambda$ (where $\lambda \neq 0$)?

Q30 [Problem/M]

$M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ represents a transformation.

Under what conditions will there be a line of invariant points passing through the Origin?

[It can in fact be shown that any line of invariant points will pass through the Origin.]

Q42 [Practice/E]

Find the equations of the invariant lines of the transformation represented by the matrix $\begin{pmatrix} 4 & 3 \\ -3 & -2 \end{pmatrix}$

Q43 [Problem/H]

Prove that invariant points of a 2D transformation always lie on a line passing through the Origin.

Q45 [Problem/H]

For the transformation matrix $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$, where a, b, c & d are positive, find a relationship between the trace $a + d$ and the determinant that must hold in order for the transformation to have an invariant line that doesn't pass through the Origin.

Q46 [Practice/M]

For the matrix $\begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$, find all of the invariant lines of the associated transformation.

Q49 [Practice/M]

Find the invariant points and lines for the transformation represented by the matrix $\begin{pmatrix} 5 & 4 \\ -4 & -3 \end{pmatrix}$.

Q52

For the transformation $\begin{pmatrix} 3 & 1 \\ 2 & k \end{pmatrix}$:

(i) For what value of k is there a line of invariant points, and find this line.

(ii) Given now that $k = 4$, find any invariant lines of the transformation.

Q53

The 2×2 matrix M represents a transformation that maps all points to the same line. In addition, points on this line are invariant points under the transformation. What conditions must apply to M?

Q56

For a 2×2 matrix $M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ with zero determinant:

(i) Find the equation of the line (which we will call the image line) to which all points are mapped.

(ii) Find the family of lines that map to specific points on the image line.

(iii) What condition must apply to a, b, c & d in order for the lines in this family to be perpendicular to the image line?

(iv) Show that if the lines in this family are invariant lines under the transformation, then $tr(M) = 1$, where $tr(M)$ (the trace of M) is defined to be $a + d$

(v) Find the condition(s) for the lines in this family to be invariant lines that are perpendicular to the image line, and give an example of a matrix satisfying these conditions.

Q57

Consider the transformation represented by the matrix

$M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$.

(i) Show that the transformation has at least one invariant line if and only if $[tr(M)]^2 \geq 4|M|$

(where $tr(M)$ (the trace of M) $= a + d$)

(ii) Show that there will be a family of invariant lines if and only if

$tr(M) = |M| + 1$

Q58

Suppose that the matrix $M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ represents a reflection in the line $y = mx$.

(i) By considering the image of a particular point under M, explain why $a^2 + b^2 = 1$.

(ii) Given that a necessary and sufficient condition for a transformation to have a line of invariant points is that $tr(M) = |M| + 1$ (where $tr(M)$ (the trace of M) $= a + d$) and by considering the image of a point on the line $y = mx$ (or otherwise), show that $M = \begin{pmatrix} \frac{1-m^2}{1+m^2} & \frac{2m}{1+m^2} \\ \frac{2m}{1+m^2} & \frac{m^2-1}{1+m^2} \end{pmatrix}$

Inverses

Q19 [Problem/M]

Prove that $\begin{pmatrix} a & c \\ b & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$

Q20 [Problem/M]

Show that if N is the left inverse of M, so that $NM = I$, then it is also the right inverse.

Q22 [Problem/M]

Assuming that $(AB)^T = B^T A^T$, prove that $(A^T)^{-1} = (A^{-1})^T$

Shears

Q31 [Problem/M]

Consider the matrix $M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$, which represents a shear. Show that it is not possible for all of the elements of the matrix to be positive. [It can be assumed that $trM = 2$.]

Q32 [Problem/M]

If the 2×2 matrix M represents a shear, what can be said about M^{-1}? [It can be assumed that the trace of a 2×2 matrix will equal 2 in the case of a shear.]

Q33 [Practice/M]

Find the invariant lines of the shear represented by the matrix
$\begin{pmatrix} 4 & -3 \\ 3 & -2 \end{pmatrix}$

Q34 [Practice/M]

Find the invariant lines of the shear represented by the matrix
$\begin{pmatrix} 7 & -4 \\ 9 & -5 \end{pmatrix}$

Q54 [Practice/M]

Show that $\frac{(a-1)^2+c^2}{c} = -\frac{b^2+(1-d)^2}{b}$ for the shear $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$.

[It can be assumed that the trace of a 2×2 matrix will equal 2 in the case of a shear.]

Simultaneous Equations

Q3 [Practice/M]

Use matrices to find the plane containing the points $(2, -1, 4), (-3, 4, 2)$ and $(1, 0, 5)$ (without using a calculator)

Q35 [Practice/E]

(i) Three planes are represented by the following equations:

$x - y + z = 1$

$2x + ky + 2z = 3$

$x + 3y + 3z = 5$

For what value of k do the planes not meet at a single point? For this value of k how are the planes configured?

(ii) If $k = 2$, find the point of intersection, using matrices.

Q36 [Practice/E]

Find the value of k for which the following equations are consistent.

$3x - 3y - z = k$

$2x - y - z = 5$

$x + 4y - 2z = 7$

Q37 [Practice/E]

Show that the following three planes meet in a line, giving the equation of that line in cartesian form.

$$\begin{aligned}
&\boldsymbol{x - y + 3z = 4} \\
&\boldsymbol{4x + 5y - 2z = 8} \\
&\boldsymbol{x + 17y - 25z = -12}
\end{aligned}$$

Q38 [Problem/E]

Consider the planes with the following equations:

$$\begin{aligned}
\boldsymbol{ax - y + z} &\boldsymbol{= 1} \\
\boldsymbol{2y - z} &\boldsymbol{= b} \\
\boldsymbol{4x + 3y - 2z} &\boldsymbol{= 2}
\end{aligned}$$

(i) Find conditions on $\boldsymbol{a}$ and $\boldsymbol{b}$ for:
(a) the 3 planes to meet at a single point
(b) the 3 planes to meet in a line
(c) no point of intersection of the 3 planes

(ii) Show that in case (c) the line of intersection of the 1st two planes is parallel to the 3rd plane.

Q39 [Practice/E]

Use matrices to find the plane containing the points $(2, -1, 4), (-3, 4, 2)$ and $(1, 0, 5)$

Q55 [Problem/M]

Given that $a + 2b = 16$ and $b - c = 2$, use a matrix argument to determine that $a + b + c = 14$

Transformations

Q1 [8 marks]

The point P is transformed by the matrix $\begin{pmatrix} 1 & 1 & 2 \\ -2 & 3 & 0 \\ 0 & 4 & -1 \end{pmatrix}$ to the image point (4, 2, –3). Find the coordinates of P.

[without using a calculator]

Q2 [Practice/M]

$M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ represents a transformation.

(i) Under what conditions will $x = 0$ be an invariant line?

(ii) Under what conditions will there be an invariant line of the form $x = \lambda$ (where $\lambda \neq 0$)?

Q17 [Problem/E]

(i) Plot the image of the unit square under the transformation represented by the matrix $\begin{pmatrix} 5 & 1 \\ 2 & 3 \end{pmatrix}$

(ii) Use (i), but with more general labels, to show that the area scale factor for the transformation $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ is $ad - bc$.

Q40 [Problem/E]

(i) Find the equation of the line that the matrix $\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$ maps all points to.

(ii) For the same transformation, find the equation of the line that maps to the point with an x-coordinate of w.

(iii) For the same transformation, for which point(s) will the x-coordinate remain unchanged by the transformation?

Q41 [Problem/M]

Derive a formula for the area of a triangle with corners at $(0,0), (a, b), (c, d)$, using matrix transformations.

Q44 [Problem/M]

Show that the matrix $\begin{pmatrix} cos2\theta & sin2\theta \\ sin2\theta & -cos2\theta \end{pmatrix}$ [representing a reflection in the line $y = tan\theta . x$] can be written as $\begin{pmatrix} \frac{1-m^2}{1+m^2} & \frac{2m}{1+m^2} \\ \frac{2m}{1+m^2} & \frac{m^2-1}{1+m^2} \end{pmatrix}$, where $m = tan\theta$

Q50 [Practice/M]

(i) Show that the transformation represented by the matrix $\begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix}$ (with determinant zero) maps all points to a particular line.

(ii) Find the line whose points all map to the point (3,1).

(iii) Without doing any calculations, what can be said about the line whose points all map to the point (6,2)?

(iv) Write down the line whose points all map to the Origin.