MAT: Specimen 2 - Q2 (4 Pages; 24/9/20)

Solution

(i) Equating coefficients:

$x^3: 0 = -a + a$

$x^2: A = b - a^2 + b$

$x: 0 = ab - ba$

$x^0: B = b^2$

So $B = b^2$ & $A = 2b - a^2$

(ii) $B = 16 = b^2$ (1)

& $A = -20 = 2b - a^2$ (2)

(2) $\Rightarrow 4b^2 = (a^2 - 20)^2$

(though there may be a spurious solution)

and then (1) $\Rightarrow 64 = a^4 - 40a^2 + 400$

$\Rightarrow a^4 - 40a^2 + 336 = 0$

$\Rightarrow a^2 = \frac{40 \pm \sqrt{1600 - 4(336)}}{2} = 20 \pm \sqrt{400 - 336} = 20 \pm 8$

So $a = \pm\sqrt{28}$ & $b = \frac{1}{2}(28 - 20) = 4$, from (2) (3)

or $a = \pm\sqrt{12}$ & $b = \frac{1}{2}(12 - 20) = -4$ (4)

These solutions all agree with (1)& (2); ie there are no spurious solutions.

So the required product is either

$\left(x^2+\sqrt{28}x+4\right)\left(x^2-\sqrt{28}x+4\right)$

or $\left(x^2+\sqrt{12}x-4\right)\left(x^2-\sqrt{12}x-4\right)$

[The official sol'ns omit the 2nd possibility, but it can easily be expanded to obtain x^4-20x^2+16]

(iii) **Method 1a**

From the factorisation $\left(x^2+\sqrt{28}x+4\right)\left(x^2-\sqrt{28}x+4\right)$,

the 1st factor gives

$x=\frac{-\sqrt{28}\pm\sqrt{28-16}}{2}=-\sqrt{7}\pm\sqrt{3}$,

and the 2nd factor gives $x=\frac{\sqrt{28}\pm\sqrt{28-16}}{2}=\sqrt{7}\pm\sqrt{3}$,

as required

Method 1b

From the factorisation $\left(x^2+\sqrt{12}x-4\right)\left(x^2-\sqrt{12}x-4\right)$,

the 1st factor gives

$x=\frac{-\sqrt{12}\pm\sqrt{12+16}}{2}=-\sqrt{3}\pm\sqrt{7}$,

and the 2nd factor gives $x=\frac{\sqrt{12}\pm\sqrt{12+16}}{2}=\sqrt{3}\pm\sqrt{7}$

as required

Method 2a

$(x+\sqrt{7}+\sqrt{3})(x-\sqrt{7}-\sqrt{3})(x+\sqrt{7}-\sqrt{3})(x-\sqrt{7}+\sqrt{3})$

$= (x^2 - (\sqrt{7}+\sqrt{3})^2)(x^2 - (\sqrt{7}-\sqrt{3})^2)$

$= x^4 + Cx^2 + D$,

where $C = -(\sqrt{7}-\sqrt{3})^2 - (\sqrt{7}+\sqrt{3})^2$

and $D = (\sqrt{7}+\sqrt{3})^2(\sqrt{7}-\sqrt{3})^2$

Then $C = -(7+3+2\sqrt{21}+7+3-2\sqrt{21}) = -20$

and $D = (7-3)^2 = 16$, as required

Method 2b

$(x+\sqrt{7}+\sqrt{3})(x+\sqrt{7}-\sqrt{3})$

$= x^2 + x(\sqrt{7}-\sqrt{3}+\sqrt{7}+\sqrt{3}) + (7-3)$

$= x^2 + \sqrt{28}x + 4$

and $(x-\sqrt{7}+\sqrt{3})(x-\sqrt{7}-\sqrt{3})$

$= x^2 + x(-\sqrt{7}-\sqrt{3}-\sqrt{7}+\sqrt{3}) + (7-3)$

$= x^2 - \sqrt{28}x + 4$

and we have already established in (ii) that

$(x^2+\sqrt{28}x+4)(x^2-\sqrt{28}x+4) = x^4 - 20x^2 + 16$

Method 2c

$(x+\sqrt{7}+\sqrt{3})(x-\sqrt{7}+\sqrt{3})$

$= x^2 + x(-\sqrt{7}+\sqrt{3}+\sqrt{7}+\sqrt{3}) - (7-3)$

$= x^2 + \sqrt{12}x - 4$

and $\left(x+\sqrt{7}-\sqrt{3}\right)\left(x-\sqrt{7}-\sqrt{3}\right)$

$= x^2 + x\left(-\sqrt{7}-\sqrt{3}+\sqrt{7}-\sqrt{3}\right) - (7-3)$

$= x^2 - \sqrt{12}x - 4$

and we know that

$\left(x^2+\sqrt{12}x-4\right)\left(x^2-\sqrt{12}x-4\right) = x^4 - 20x^2 + 16$