MAT: Specimen 1 - Q5 (3 Pages; 24/9/20)

(i) [The wording seems to be slightly ambiguous: presumably it means that a y is added after the two repetitions (rather than requiring that the last note of the (2nd) repetition is a y).]

From I & II, xy is a song, and then from III, yx is also a song.

Applying II to these two songs gives: $xxyxyy$ & $xyxyxy$

And, from III we also get $yyxyxx$ & $yxyxyx$

(ii) The 1st part follows from II & III (noting that there are no other ways of creating songs of length $2m+2$). Strictly speaking, we should also check that none of these songs are the same: there will be k different songs resulting from applying II, and then the further k songs resulting from applying III will be new ones, because they all end in an x (whereas the 1st batch of k songs all end in a y).

For the 2nd part:

To clarify: $n \rightarrow 2^{n+1}-2$ (length) $\rightarrow 2^n$ (number of songs of that length); ie not all lengths will be possible.

This can be tackled by induction:

First of all, we show that the result is true for $n=1$:

There are $2^1=2$ songs of length $2^2-2=2$ (namely xy & yx).

[Note that the natural numbers start at 1 (this is the usual definition; in some countries, they include 0); the question presumably meant to say "... for each natural number n"]

Then assume that there are 2^k songs of length $2^{k+1}-2$ (*)

rtp [result to prove]: there are 2^{k+1} songs of length $2^{k+2}-2$ (**)

1st part of (ii) & (*) ⇒ there are $2(2^k)$ songs of length

$2(2^{k+1}-2)+2=2^{k+2}-2$, as required

If the result is true for $n=1$, then from (**) it will be true for

$n=2,3,\dots$ and hence all n, by the principle of induction.

Alternative method:

As n is increased by 1, the new length L_n is related to the previous one by $L_n=2L_{n-1}+2$ and the number of songs is multiplied by 2 (from the 1st part of (ii)).

So we see that

$L_1=2,\ L_2=2^2+2,\ L_3=2(2^2+2)+2=2+2^2+2^3$,

and so on, giving

$L_n=2+2^2+2^3+\dots+2^n=\frac{2(2^n-1)}{2-1}=2^{n+1}-2$, as required.

(iii) [Remember that MAT questions don't usually involve anything too obscure, so it's worth considering simple outcomes.]

For the songs from the "Classical period", the possible lengths were governed entirely by the $2m+2$ rule, and each possible length only gave rise to another length of higher value. With the "later period" songs, we can now go backwards as well - potentially making the situation very complicated.

However, we note that:

(a) For any given length, we can always produce a length of greater value.

(b) To cover any missed values, we can always subtract one: since this is allowed by IV if the last note is a y; and if it is an x, then we can obtain a song of the same length, ending in a y, by III - and then subtract one from that.

ie we can make a song of any length.