MAT: Specimen 1 - Q4 (2 pages; 28/11/20)

(i) **1st part**

Area of $ABC = \frac{1}{2}\ AC.BQ$ (where Q is the foot of the perpendicular from B to CA)

$= \frac{1}{2}b(ABsin\alpha) = \frac{1}{2}bcsin\alpha$

2nd part

Similarly, Area of $ABC = \frac{1}{2}acsin\beta$ and Area of $ABC = \frac{1}{2}absin\gamma$

And so $\frac{1}{2}bcsin\alpha = \frac{1}{2}acsin\beta = \frac{1}{2}absin\gamma$

which gives $bsin\alpha = asin\beta$, $csin\alpha = asin\gamma$ and $csin\beta = bsin\gamma$,

and hence $\frac{b}{sin\beta} = \frac{a}{sin\alpha}$, $\frac{c}{sin\gamma} = \frac{a}{sin\alpha}$ (and $\frac{c}{sin\gamma} = \frac{b}{sin\beta}$) ;

ie $\frac{a}{sin\alpha} = \frac{b}{sin\beta} = \frac{c}{sin\gamma}$

(ii) The form of the answer suggests subtracting Area(AQR)+Area(BPR)+Area(CQP) from Area(ABC), and symmetry suggests that we should try to show that

Area(AQR) $= cos^2\alpha\ \times Area(ABC)$

Now Area(AQR) $= \frac{1}{2}AQ.ARsin\alpha$,

Considering the right-angled triangle ABQ,

[as we need to incorporate the fact that Q is the foot of the perpendicular from B onto AC]

$AQ = ABcos\alpha = ccos\alpha$,

and also, $AR = ACcos\alpha = bcos\alpha$

So Area(AQR) $= \frac{1}{2}(ccos\alpha)(bcos\alpha)sin\alpha$

and hence $\frac{Area(AQR)}{Area\ (ABC)} = \frac{\frac{1}{2}bccos^2\alpha sin\alpha}{\frac{1}{2}bcsin\alpha} = cos^2\alpha,$ as required.

(iii) In order that Area(PQR) = 0, P, Q & R must lie on a straight line. A bit of experimenting shows that ABC has to be right-angled for this to happen.