MAT Exercises – Trigonometry - Sol'ns

(16 pages; 4/11/22)

(1) How many solutions does the equation

$sin(2cos(2x) + 2) = 0$ have, for $0 \leq x \leq 2\pi$?

Solution

With $u = 2cos(2x) + 2, 0 \leq x \leq 2\pi \Rightarrow 2(-1) + 2 \leq u \leq 2(1) + 2$

ie $0 \leq u \leq 4$

Then $sinu = 0 \Rightarrow u = 0\ or\ \pi$

$\Rightarrow \cos(2x) = -1\ or\ \frac{\pi-2}{2} = \frac{\pi}{2} - 1$

Now making the substitution $w = 2x, 0 \leq w \leq 4\pi$

Referring to the graph of $cosw$,

$cosw = -1$ has 2 solutions (for w), and $cosw = \frac{\pi}{2} - 1$ has 4 solutions; making 6 solutions in total.

As $x = \frac{w}{2}$, there are also 6 solutions for x.

[A variation on the above approach is to say that

$2cos(2x) + 2$ must equal $n\pi$, for suitable integer n

Then, either $n = 0$, with $cos(2x) = -1$,

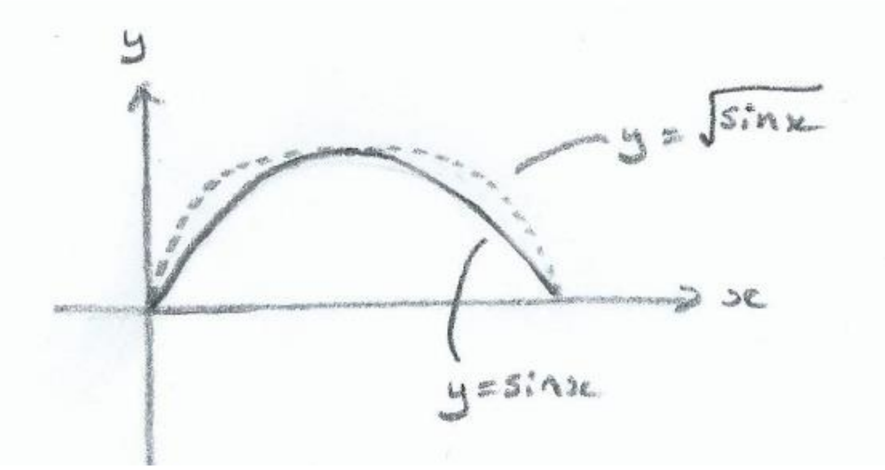
or $n = 1$, with $cos(2x) = \frac{\pi}{2} - 1$

(no other values of n are consistent with $2cos(2x) + 2$),

as before.]

(2) Sketch (i) $y = \sqrt{sinx}$ and (ii) $y = (sinx)^{\frac{1}{n}}$ for large positive integer n (for $0 \leq x \leq \pi$ in both cases).

Solution



(i) Note that, for $0 < y < 1, \sqrt{y} > y$

So, for $y = \sqrt{sinx}$, the graph will hug the $y - axis$ more than for $y = sinx$.

Also, if $f(x) = \sqrt{sinx}$, $f'(x) = \frac{1}{2}(sinx)^{-\frac{1}{2}} cosx$,

so that $f'(0) = \infty$ (strictly speaking, it is 'undefined');

ie the graph is vertical at $x = 0$ (and also $x = \pi$, by symmetry).

(ii) The effect is greater for larger n, and the graph tends to a rectangular shape.

(3) What is the period of $2\,sin\left(3x+\frac{\pi}{4}\right)+3cos\,(\frac{2x}{3}-\frac{\pi}{3})$?

Solution

The period T_1 of $2\,sin\left(3x+\frac{\pi}{4}\right)$ satisfies $3T_1 = 2\pi$

[as $2sin\left(3[0]+\frac{\pi}{4}\right) = 2sin\left(2\pi+\frac{\pi}{4}\right)$]; ie $T_1 = \frac{2\pi}{3}$

Similarly for $3cos\,(\frac{2x}{3}-\frac{\pi}{3})$, $\frac{2T_2}{3} = 2\pi$, so that $T_2 = 3\pi$

The period of the sum of these functions is the LCM of these two periods; ie 6π.

(4) Assuming that $sin^2\theta + cos^2\theta = 1$, but without using any compound angle results, show that $sin\theta cos\theta \leq \frac{1}{2}$

Solution

$(sin\theta - cos\theta)^2 \geq 0 \Rightarrow sin^2\theta + cos^2\theta - 2sin\theta cos\theta \geq 0$

$\Rightarrow 1 \geq 2sin\theta cos\theta \Rightarrow sin\theta cos\theta \leq \frac{1}{2}$

(5) Solve $\sin(2\theta - \frac{\pi}{6}) = 0.5 \ \ (0 < \theta < 2\pi)$

Solution

Let $= 2\theta - \frac{\pi}{6}$, so that $-\frac{\pi}{6} < u < 4\pi - \frac{\pi}{6}$

Then $sin\, u = 0.5 \Rightarrow u = \frac{\pi}{6}, \frac{\pi}{6} + 2\pi$ and $\pi - \frac{\pi}{6}, \pi - \frac{\pi}{6} + 2\pi$

ie $u = \frac{\pi}{6}, \frac{13\pi}{6}, \frac{5\pi}{6}$ & $\frac{17\pi}{6}$ or $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}$ & $\frac{17\pi}{6}$

so that $\theta = \frac{1}{2}\left(u + \frac{\pi}{6}\right) = \frac{2\pi}{12}, \frac{6\pi}{12}, \frac{14\pi}{12}$ & $\frac{18\pi}{12}$

ie $\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{7\pi}{6}$ & $\frac{3\pi}{2}$

(6) Solve $sin\theta = cos4\theta$ for $0 < \theta < \pi$

Solution

$sin\theta = sin(\frac{\pi}{2} - 4\theta)$

Hence $\theta = \frac{\pi}{2} - 4\theta + 2n\pi$ (1) or $\theta = \left(\pi - \left[\frac{\pi}{2} - 4\theta\right]\right) + 2n\pi$ (2)

From (1), $5\theta = \frac{\pi(1+4n)}{2}$, so that $\theta = \frac{\pi(1+4n)}{10}$

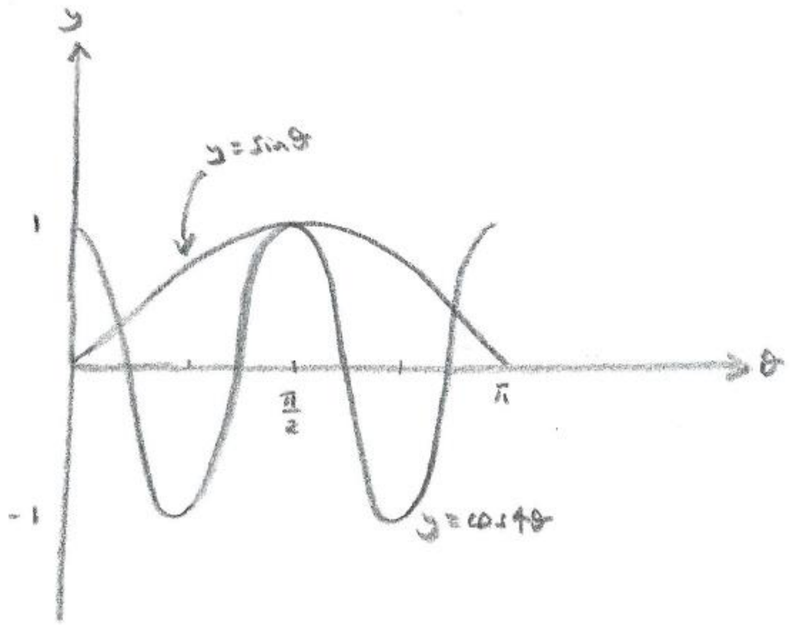
giving $\theta = \frac{\pi}{10}$, $\frac{\pi}{2}$ or $\frac{9\pi}{10}$

From (2), $-3\theta = \frac{\pi(1+4n)}{2}$, so that $\theta = \frac{-\pi(1+4n)}{6}$

giving $\theta = \frac{\pi}{2}$ again

Thus, the solutions are $\theta = \frac{\pi}{10}$, $\frac{\pi}{2}$ or $\frac{9\pi}{10}$

A sketch confirms that these are plausible.



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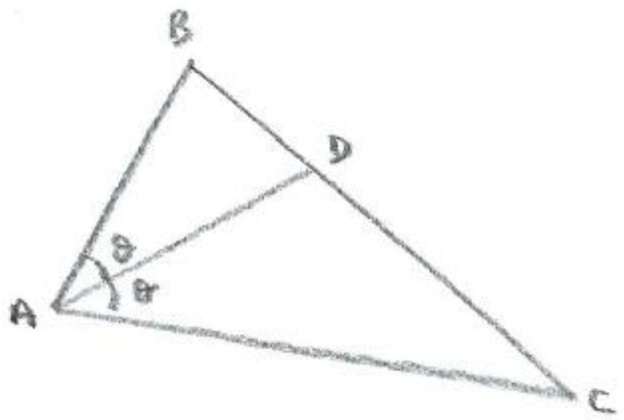
(8) Angle Bisector Theorem

Referring to the diagram below, the Angle Bisector theorem says that $\frac{BD}{DC} = \frac{AB}{AC}$. Prove the Angle Bisector Theorem.



Solution

By the Sine rule for triangle ABD, $\frac{BD}{sin\theta} = \frac{AB}{sinADB}$ (1)

and, for triangle ADC, $\frac{DC}{sin\theta} = \frac{AC}{sinADC} = \frac{AC}{sinADB}$ (2)

Then (1) $\Rightarrow \frac{sin\theta}{sinADB} = \frac{BD}{AB}$ and (2) $\Rightarrow \frac{sin\theta}{sinADB} = \frac{DC}{AC}$

so that $\frac{BD}{AB} = \frac{DC}{AC}$

and hence $\frac{BD}{DC} = \frac{AB}{AC}$