MAT Exercises – Series - Sol'ns (3 pages; 2/11/22)

(1) Show that $\sum_{r=0}^{n} \binom{n}{r} = 2^n$

Solution

Method 1: Consider $(1+1)^n$

Method 2: Pascal's triangle

The sum of each row is twice the sum of the previous one.

eg $1+5+10+10+5+1$

$= (1+10+5)[alternate\ terms] + (5+10+1)$

$= 2(1+10+5) = 2(1+[4+6]+[4+1])$

& $1+6+15+20+15+6+1$

$= (1+15+15+1) + (6+20+6)$

$= (1+[5+10]+[10+5]+1)$

$+([1+5]+[10+10]+[5+1])$

Method 3: Counting ways of selecting any number of items

1st counting method: $\sum_{r=0}^{n} \binom{n}{r}$

2nd counting method: For each object, there are 2 choices: include or exclude; giving 2^n

[Note: 1 way of choosing no objects is included in the total.]

Method 4: Induction

If true for $n = k$, so that $\sum_{r=0}^{k} \binom{k}{r} = 2^k$,

then $\sum_{r=0}^{k+1} \binom{k+1}{r} = \binom{k+1}{0} + \{\sum_{r=1}^{k} \binom{k+1}{r}\} + \binom{k+1}{k+1}$

$= 1 + \sum_{r=1}^{k} \left\{ \binom{k}{r-1} + \binom{k}{r} \right\} + 1$

$= 1 + \{\sum_{r-1=0}^{k-1} \binom{k}{r-1}\} + [\{\sum_{r=0}^{k} \binom{k}{r}\} - \binom{k}{0}] + 1$

$$= 1 + \{\textstyle\sum_{R=0}^{k-1} \binom{k}{R}\} + [2^k - 1] + 1$$

$$= 1 + \{\textstyle\sum_{R=0}^{k} \binom{k}{R}\} - \binom{k}{k} + 2^k$$

$$= 1 + 2^k - 1 + 2^k$$

$$= 2^{k+1}$$