MAT Exercises – Integers - Sol'ns (8 pages; 3/11/22)

(1) Can n^3 equal $n + 12345670$ (where n is a positive integer)?

Solution

Rearrange to $n^3 - n = 12345670$

$n^3 - n = n(n^2 - 1) = n(n-1)(n+1)$, and one of these factors must be a multiple of 3; whereas 12345670 is not a multiple of 3 (since $1 + 2 + 3 + 4 + 5 + 6 + 7 + 0$ isn't a multiple of 3); so answer is No.

(2) Find all positive integer solutions of the equation

$xy - 8x + 6y = 90$

Solution

[Aiming for something of the form $f(x)g(y) = c$, where c is an integer:]

$xy - 8x + 6y = (x + 6)(y - 8) + 48$,

so that the original equation is equivalent to

$(x + 6)(y - 8) = 42$

The positive integer solutions are given by:

$x + 6 = 7, y - 8 = 6$

$x + 6 = 14, y - 8 = 3$

$x + 6 = 21\ y - 8 = 2$

$x + 6 = 42, y - 8 = 1$,

so that the solutions are:

$x = 1, y = 14$

$x = 8, y = 11$

$x = 15, y = 10$

$x = 36, y = 9$

(3) Show that $3^{57} - 2^{57}$ cannot be prime.

Solution

We could consider using the result

$x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})$

but it isn't of any use having $x - y = 3 - 2 = 1$.

However, we can write $3^{57} - 2^{57}$ as $(3^{19})^3 - (2^{19})^3$, for example, to give the factor $3^{19} - 2^{19}$ (or writing it instead as

$(3^3)^{19} - (2^3)^{19}$, $3^3 - 2^3$ is also seen to be a factor).

So $3^{57} - 2^{57}$ isn't a prime number.

(4) Prove that there are no positive integers m and n such that
$m^2 = n^2 + 1$

Solution

[Proof by contradiction]

Suppose that $m^2 = n^2 + 1$, where m and n are positive integers.

Then $m^2 - n^2 = 1$,

and hence $(m - n)(m + n) = 1$

As m and n are integers, $m - n$ and $m + n$ will also be integers, and so they are either both 1 or both -1

But $m + n > 0$, so that $m - n = 1$ and $m + n = 1$

Subtracting the 1st eq'n from the 2nd gives $2n = 0$, so that $n = 0$, which contradicts the assumption that n is a positive integer.

So there are no positive integers m and n such that $m^2 = n^2 + 1$