MAT Exercises – Inequalities - Sol'ns (19 pages; 4/11/22)

(1) Are the following true or false?

(i) $a < b \Rightarrow \frac{1}{a} > \frac{1}{b}$

(ii) $a < b \Rightarrow a^2 < b^2$

(iii) $a < b$ & $c < d \Rightarrow a + c < b + d$

(iv) $a < b$ & $c < d \Rightarrow a - c < b - d$

Solution

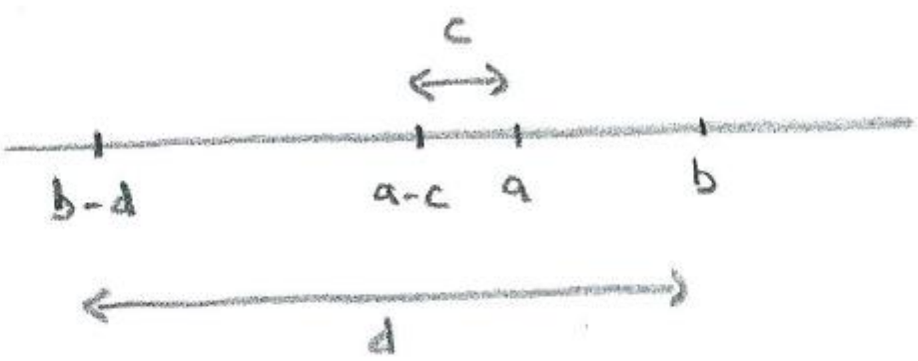
(i) Not true if $a < 0$ & $b > 0$ (consider the graph of $y = 1/x$)

(ii) Not true if $a < 0$ & $b < 0$ or

if $a < 0, b > 0$ & $|b| < |a|$(consider the graph of $y = x^2$)

(iii) True: $a < b \Rightarrow a + c < b + c < b + d$

(iv) False: For example, $8 < 9$ and $2 < 4$, but it is not true that $8 - 2 < 9 - 4$; see diagram



(2) Assuming that $sin^2\theta + cos^2\theta = 1$, but without using any compound angle results, show that $sin\theta cos\theta \leq \frac{1}{2}$

Solution

$(sin\theta - cos\theta)^2 \geq 0 \Rightarrow sin^2\theta + cos^2\theta - 2sin\theta cos\theta \geq 0$

$\Rightarrow 1 \geq 2sin\theta cos\theta \Rightarrow sin\theta cos\theta \leq \frac{1}{2}$

(3) Which is larger: $\frac{\sqrt{7}}{2}$ or $\frac{1+\sqrt{6}}{3}$ (without using a calculator)?

Solution

Considering the difference of squares:

$\frac{7}{4} - \frac{(1+2\sqrt{6}+6)}{9} = \frac{63-28-8\sqrt{6}}{36} > \frac{35-8(3)}{36} > 0$; so $\frac{\sqrt{7}}{2}$ is larger

[Another approach is to investigate $\frac{\left(\frac{7}{4}\right)}{\left(\frac{7+2\sqrt{6}}{9}\right)} = \frac{63(7-2\sqrt{6})}{4(49-24)} = \frac{63(7-2\sqrt{6})}{100}$, but it isn't as easy to show that this expression is greater than 1]

(4) Is $\frac{6}{7} < \frac{2}{\sqrt{5}}$?

Solution

$\frac{6}{7} < \frac{2}{\sqrt{5}} \Leftrightarrow \frac{36}{49} < \frac{4}{5}$

$49 \times 0.8 = \frac{1}{10}(320 + 72) = 39.2 > 36$

So $\frac{36}{49} < \frac{39.2}{49} = 0.8 = \frac{4}{5}$

Answer is Yes.

(5) Is $log_2 3 > \frac{3}{2}$?

Solution

$log_2 3 > \frac{3}{2} \Leftrightarrow 3 > 2^{\frac{3}{2}}$ (as $y = 2^x$ is an increasing function)

$\Leftrightarrow 3^2 > 2^3$

So answer is Yes.

(6) The probability that a (biased) coin shows Heads is p, and the probability that it shows Tails is q. Prove that $pq \leq \frac{1}{4}$.

Solution

$pq \leq \frac{1}{4} \Leftrightarrow 4p(1-p) \leq 1$ (as $p + q = 1$)

$\Leftrightarrow 4p^2 - 4p + 1 \geq 0$

As LHS $= 4(p - \frac{1}{2})^2$, the result is proved.

(7) Show that if $X > 1$ & $Y > 1$, then $X + Y < XY + 1$

Solution

$X + Y < XY + 1 \Leftrightarrow X + Y - XY - 1 < 0$

$\Leftrightarrow X(1 - Y) + Y - 1 < 0$

$\Leftrightarrow (X - 1)(1 - Y) < 0$

$\Leftrightarrow (X - 1)(Y - 1) > 0$

Then $X > 1$ & $Y > 1 \Rightarrow (X - 1)(Y - 1) > 0 \Rightarrow X + Y < XY + 1$

(8) Show that $e^3 > 4e^{\frac{3}{2}}$

Solution

An equivalent result to prove is $e^{\frac{3}{2}} > 4$ (dividing both sides by $e^{\frac{3}{2}}$, which is positive)

$\Leftrightarrow e^3 > 16$ (as the function $y = x^2$ is increasing for $x > 0$)

$e^3 > (2 + 0.7)^3 > 2^3 + 3(2^2)(0.7) = 8 + 8.4 > 16$,

so that the original result is also true

(9) Let x, y & z be positive real numbers.

(i) If $x + y \geq 2$, is it necessarily true that $\frac{1}{x} + \frac{1}{y} \leq 2$?

(ii) If $x + y \leq 2$, is it necessarily true that $\frac{1}{x} + \frac{1}{y} \geq 2$?

Solution

(i) No: if x (say) is very small, then $\frac{1}{x}$ will be very large.

(ii) Note that, when $x = y = 1, \frac{1}{x} + \frac{1}{y} = 2$

Also, if the result is true for $x + y = 2$, then if $x\ or\ y$ is made smaller, so that $x + y < 2, \frac{1}{x} + \frac{1}{y}$ becomes larger, so that the result is still true. So, WLOG (without loss of generality), we need only investigate the case where $x + y = 2$.

Experimenting with some numbers, we get the impression that $\frac{1}{x} + \frac{1}{y} \geq 2$. So, aiming for a proof by contradiction, suppose that $\frac{1}{x} + \frac{1}{y} < 2$

Then, $\frac{x+y}{xy} < 2$, so that $2 < 2x(2 - x)$ [as $xy > 0$]

and hence $1 < 2x - x^2$ and $x^2 - 2x + 1 < 0\ \ or\ \ (x - 1)^2 < 0$,

which is impossible.

Thus $\frac{1}{x} + \frac{1}{y} \geq 2$ when $x + y \leq 2$

Alternative approach

To prove that $\frac{1}{x} + \frac{1}{y} \geq 2$ when $x + y = 2$,

we note that WLOG we need only consider solutions of the form $x = 1 + \delta, y = 1 - \delta$ (where $\delta > 0$).

But the reduction from $\frac{1}{1}$ to $\frac{1}{1+\delta}$ will be outweighed by the rise from $\frac{1}{1}$ to $\frac{1}{1-\delta}$ [consider the extreme cases $\frac{1}{1000}\ to\ \frac{1}{1001}$ versus $\frac{1}{4}\ to\ \frac{1}{3}$, which shows that the change of 1 in the denominator has a greater effect when the denominator is smaller, as it is with $1 - \delta$, compared to $1 + \delta$]