2022 MAT – Q3 (4 pages; 6/11/23)

Solution

(i) The curves intersect (or touch) where $(x^2-1)^2=(x^2-1)^3$

$\Leftrightarrow (x^2-1)^2(1-[x^2-1])=0$

$\Leftrightarrow (x^2-1)^2(2-x^2)=0$

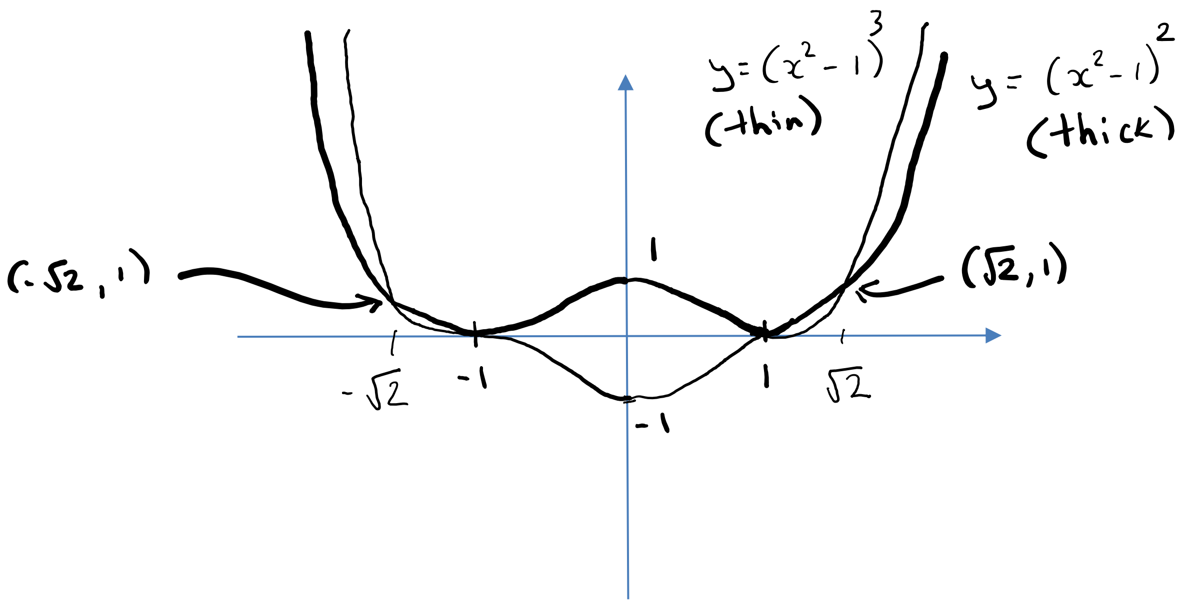
$\Leftrightarrow x=\pm 1$ (touching) or $\pm\sqrt{2}$ (crossing)

The curve $y=(x^2-1)^2$ touches the x axis when $x=\pm 1$, and crosses the y axis when $y=1$.

The curve $y=(x^2-1)^3$ touches the x axis when $x=\pm 1$, and crosses the y axis when $y=-1$.

$y=(x^2-1)^2=(x-1)^2(x+1)^2$

and $y=(x^2-1)^3=(x-1)^3(x+1)^3$



(ii) $(x^2-1)^n \geq 0$ for even n,

and for $a > 0$, $(x^2-1)^n > 0$ for $0 \leq x \leq a$, except for $x = 1$ (if $a \geq 1$)

So the integrand is positive for a finite region in $[0, a]$ and never negative. Hence the integral is positive, and so cannot equal zero.

(iii) We see that a_m must be greater than 1, otherwise the integrand would be negative over the whole range – except at the single point $x = 1$.

And $\int_0^1 (x^2-1)^{2m-1} dx < 0$

As a is increased beyond 1, $\int_1^a (x^2-1)^{2m-1} dx > 0$ and increases (from zero) without limit as a increases, and can therefore attain any positive value.

Hence, when $\int_1^a (x^2-1)^{2m-1} dx = -\int_0^1 (x^2-1)^{2m-1} dx$,

$\int_0^a (x^2-1)^{2m-1} dx = 0$, and then $a_m = a$

(iv) $\int_0^{a_1} (x^2-1)\, dx = 0 \Rightarrow \left[\frac{1}{3}x^3 - x\right]_0^{a_1} = 0$

$\Rightarrow \frac{1}{3}{a_1}^3 - a_1 = 0$

Then, as $a_1 > 0$, $a_1 = \sqrt{3}$

(v) $\int_0^{a_2} (x^2-1)^3 dx = 0$

$\Rightarrow \int_0^{a_2} x^6 - 3x^4 + 3x^2 - 1\, dx = 0$

$\Rightarrow \left[\frac{1}{7}x^7 - \frac{3}{5}x^5 + x^3 - x\right]_0^{a_2} = 0$

$\Rightarrow A = \frac{1}{7}{a_2}^7 - \frac{3}{5}{a_2}^5 + {a_2}^3 - a_2 = 0$

When $a_2 = \sqrt{2}, A = \frac{1}{7}\left(8\sqrt{2}\right) - \frac{3}{5}\left(4\sqrt{2}\right) + 2\sqrt{2} - \sqrt{2}$

$= \frac{\sqrt{2}}{35}(40 - 84 + 35) < 0$

When $a_2 = \sqrt{3}, A = \frac{1}{7}\left(27\sqrt{3}\right) - \frac{3}{5}\left(9\sqrt{3}\right) + 3\sqrt{3} - \sqrt{3}$

$= \frac{\sqrt{3}}{35}(135 - 189 + 70) = \frac{\sqrt{3}}{35}(16) > 0$

Thus there is a change of sign of A, and hence $\sqrt{2} < a_2 < \sqrt{3}$ (as the (continuous) graph of A must cross the x-axis between $\sqrt{2}$ & $\sqrt{3}$).

(vi) From the given result, $\sqrt{2} < a_m$

And from the graph of $(x^2 - 1)^3$ in (i), we can see that

$\int_0^1 (x^2 - 1)^{2m-1} dx > -1$;

ie the negative contribution to $I = \int_0^a (x^2 - 1)^{2m-1} dx$ is limited (for fixed $a > \sqrt{2}$).

The contribution to I from $\int_1^{\sqrt{2}} (x^2 - 1)^{2m-1} dx$ reduces as m increases, but is always positive.

Meanwhile the contribution to I from $\int_{\sqrt{2}}^a (x^2 - 1)^{2m-1} dx$ increases without limit, as as m increases.

And so, as m increases, $\int_{\sqrt{2}}^a (x^2 - 1)^{2m-1} dx$ will balance the reducing contributions from $\int_0^1 (x^2 - 1)^{2m-1} dx$ and

$\int_1^{\sqrt{2}}(x^2-1)^{2m-1}dx$, for increasingly smaller a.

Thus a_m decreases as m increases, and so, as $\sqrt{2} < a_m$,

it follows that the limiting value of a_m is $\sqrt{2}$;

ie this is the approximate value of a_m for very large m.