2022 MAT – Q2 (2 pages; 2/11/23)

Solution

(i) If $x^2 - 19y^2 = z$, then $z^2 = (x^2 - 19y^2)^2$

$= (x^2 + 19y^2)^2 - 4x^2(19y^2)$

$= (x^2 + 19y^2)^2 - 19(2xy)^2$, so that $N = 19$

(ii) **1st Part**

If $x = 13$ & $y = 3$, then $z = 13^2 - 19(3)^2 = 169 - 171 = -2$

2nd Part

From (i), $z^2 = (x^2 + 19y^2)^2 - 19(2xy)^2$,

so that $4 = (169 + 171)^2 - 19(78)^2$

Thus the required x & y are 340 and 78.

(iii) **1st Part**

$4 = (340)^2 - 19(78)^2 \Rightarrow 1 = 170^2 - 19(39)^2$,

so that the required x & y are 170 and 39.

2nd Part

From (i), when $x^2 - 19y^2 = z$, then

$z^2 = (x^2 + 19y^2)^2 - 19(2xy)^2$,

so that $1^2 = (170^2 + 19(39)^2)^2 - 19(2(170)(39))^2$

Thus another solution to $x^2 - 19y^2 = 1$

is $x = 170^2 + 19(39)^2,\ y = 2(170)(39)$

(iv) $x^2 - 25y^2 = 1 \Rightarrow (x - 5y)(x + 5y) = 1$

Then, as x & y are whole numbers,

either $x - 5y = 1$ & $x + 5y = 1$ (A)

or $x - 5y = -1$ & $x + 5y = -1$ (B)

Adding the equations in (A) gives $2x = 2$, so that $x = 1$

Adding the equations in (B) gives $2x = -2$, so that $x = -1$

Thus there are no sol'ns with $x > 1$.

(v) [Using the same idea as in (i)]

If $x^2 - 17y^2 = z$, then $z^2 = (x^2 - 17y^2)^2$

$= (x^2 + 17y^2)^2 - 4x^2(17y^2)$

$= (x^2 + 17y^2)^2 - 17(2xy)^2$ (*)

[Note that we can't start with $x = 1, y = 0, z = 1$, as this only leads to a 'new' sol'n of $x = 1, y = 0$]

[In (ii), we were given initial values of $x = 13, y = 3$; presumably the corresponding values with 17 in place of 19 are intended to be fairly easy to find.]

If $y = 1$, then $x^2 - 17y^2 = x^2 - 17$, and we could try $x = 4$,

as this gives $x^2 - 17y^2 = -1$

and hence, from (*), $(-1)^2 = (x^2 + 17y^2)^2 - 17(2xy)^2$,

so that a further sol'n of $x^2 - 17y^2 = 1$ is

$x = 4^2 + 17(1)^2 = 33$ & $y = 2(4)(1) = 8$

[As it turned out, the method wasn't quite the same as before, as we didn't need to divide through by anything as in (iii).]