# 2021 MAT – Q3 (3 pages; 6/11/23)

## Solution

(i) The point (0,0) lies on the graph, and so $p(0) = 0$.

At a turning point, the gradient is zero (by definition), and so $p'(0) = 0$.

Because (0,0) is a turning point, the graph of $y = p(x)$ touches the $x$-axis ($y = 0$) at (0,0).

This means that there are repeated roots of the equation

$p(x) = 0$ when $x = 0$; ie two of the factors of $p(x)$ are $(x - 0)$;

ie $p(x) = x^2 g(x)$, for some polynomial $g(x)$, as required.

[Alternatively:

As $p(0) = 0$, $(x - 0)$ is a factor of $p(x)$, by the Factor theorem.

So $p(x) = xh(x)$, for some polynomial $h(x)$.

Then, by the Chain rule, $p'(x) = h(x) + xh'(x)$

and so, $p'(0) = h(0) + 0$,

and therefore $0 = h(0)$.

Then, by the Factor theorem, $(x - 0)$ is a factor of $h(x)$,

so that $h(x) = xg(x)$, for some polynomial $g(x)$,

and hence $p(x) = x^2 g(x)$.]

(ii) $r(x) = (x - a)^2 k(x)$

Proof: Let $y = R(x)$ be the graph obtained by translating

$y = r(x)$ by an amount $a$ to the left. This graph has a turning point at (0,0), so that $R(x) = x^2 g(x)$.

Translating $y = R(x)$ by an amount $a$ to the right, to give

$y = r(x)$, is achieved by replacing $x$ by $x - a$,

so that $r(x) = (x - a)^2 g(x - a)$, or $r(x) = (x - a)^2 k(x)$,

after $g(x - a) = a_1(x - a)^n + a_2(x - a)^{n-1}$ ... is expanded to give a polynomial of the form $k(x) = b_1 x^n + b_2 x^{n-1} + \cdots$

(iii)(a) From (ii), $f(x) = (x - a)^2 g_1(x)$

and $f(x) = (x - [-a])^2 g_2(x) = (x + a)^2 g_2(x)$

Thus both $(x - a)^2$ & $(x + a)^2$ are factors of $f(x)$,

and so $f(x) = k(x - a)^2 (x + a)^2$, as $f(x)$ is a quartic.

(b) There is symmetry about the $y$-axis.

Proof: $f(-x) = k(-x - a)^2 (-x + a)^2$

$= k(x + a)^2 (a - x)^2$

$= k(x + a)^2 (x - a)^2 = f(x)$

(c) $x = 0$ (by symmetry)

(iv) Any such polynomial must be of the form

$f(x) = kx^2 (x - 2)^2$ (in order to have turning points at

$(0, 0)$ & $(2, 0)$)

In order for there to be a turning point at $(1, 3)$, we require that

$f(1) = 3$ (so that $k = 3$) and $f'(1) = 0$

Now, $f'(x) = 3(2x)(x-2)^2 + 3x^2(2)(x-2)$

and so $f'(1) = 6 + (-6) = 0$

So the required polynomial does exist.

[The Official sol'ns use (iii), with $a = 1$, and translate 1 to the right.]

(v) In order for there to be turning points at $(1, 6)$ and $(4, 6)$, the

polynomial must have the form $f(x) = a(x-1)^2(x-4)^2 + 6$

Then $f(2) = 3 \Rightarrow 4a + 6 = 3 \Rightarrow a = -\frac{3}{4}$

Now, $f'(x) = 2a(x-1)(x-4)^2 + 2a(x-1)^2(x-4)$

so that $f'(2) = 2a(4) + 2a(-2) = 4a \neq 0$

Thus no such polynomial exists.

[The Official sol'ns employs a proof by contradiction, translating

the supposed polynomial so as to produce turning points at $\pm a$,

so that the remaining turning point has to lie at $x = 0$, from (iii).]