2021 MAT – Q2 (3 pages; 18/10/22)

Solution

(i) $\ln\left(1-\frac{1}{2}\right)=-\frac{1}{2}-\frac{\left(\frac{1}{2}\right)^2}{2}-\frac{\left(\frac{1}{2}\right)^3}{3}-\frac{\left(\frac{1}{2}\right)^4}{4}-\frac{\left(\frac{1}{2}\right)^5}{5}-\cdots$

$\Rightarrow \ln(2^{-1})[=-\ln(2)]=-(\frac{1}{2}+\frac{1}{2\times2^2}+\frac{1}{3\times2^3}+\frac{1}{4\times2^4}+\frac{1}{5\times2^5}+\cdots)$

$\Rightarrow \ln 2=\frac{1}{2}+\frac{1}{2\times2^2}+\frac{1}{3\times2^3}+\frac{1}{4\times2^4}+\frac{1}{5\times2^5}+\cdots$, as required

(ii) $\ln 2<\frac{1}{2}+\frac{1}{2\times2^2}+\frac{1}{3\times2^3}+\frac{1}{3\times2^4}+\frac{1}{3\times2^5}+\cdots$

$=\frac{1}{2}+\frac{1}{8}+\frac{1}{3\times2^3}\cdot\frac{1}{1-\frac{1}{2}}$ (from the GP with 1st term $\frac{1}{3\times2^3}$ and common ratio $\frac{1}{2}$)

$=\frac{5}{8}+\frac{1}{12}=\frac{17}{24}$

Also, $\ln 2-\frac{16}{24}=-\frac{2}{3}+\frac{1}{2}+\frac{1}{2\times2^2}+\frac{1}{3\times2^3}+\frac{1}{4\times2^4}+\frac{1}{5\times2^5}+\cdots$

$=-\frac{1}{6}+\frac{1}{8}+\frac{1}{24}+\frac{1}{64}+\frac{1}{160}+\cdots$

$=\frac{-4+3+1}{24}+\frac{1}{64}+\frac{1}{160}+\cdots$

$=\frac{1}{64}+\frac{1}{160}+\cdots>0$

So $\frac{16}{24}<\ln 2<\frac{17}{24}$; ie $k=16$

(iii) $\ln\left(\frac{3}{2}\right)=\ln\left(1-[-\frac{1}{2}]\right)$

$=-(-\frac{1}{2})-\frac{\left(-\frac{1}{2}\right)^2}{2}-\frac{\left(-\frac{1}{2}\right)^3}{3}-\frac{\left(-\frac{1}{2}\right)^4}{4}-\frac{\left(-\frac{1}{2}\right)^5}{5}-\cdots$

$= \frac{1}{2} - \frac{1}{2\times2^2} + \frac{1}{3\times2^3} - \frac{1}{4\times2^4} + \frac{1}{5\times2^5} - \cdots$, as required.

Then $\ln\left(\frac{3}{2}\right) = ln3 - ln2$,

so that, from (i), $ln3 = \{\frac{1}{2} + \frac{1}{2\times2^2} + \frac{1}{3\times2^3} + \frac{1}{4\times2^4} + \frac{1}{5\times2^5} + \cdots\}$

$+\{\frac{1}{2} - \frac{1}{2\times2^2} + \frac{1}{3\times2^3} - \frac{1}{4\times2^4} + \frac{1}{5\times2^5} - \cdots\}$

$= 1 + \frac{2}{3\times2^3} + \frac{2}{5\times2^5} + \cdots$

$= 1 + \frac{1}{3\times2^2} + \frac{1}{5\times2^4}(+\frac{1}{7\times2^6}) + \cdots$, as required

(iv) $1 + \frac{1}{3\times2^2} = \frac{13}{12}$, so that $ln3 > \frac{13}{12}$

And $\frac{1}{n\times2^{n-1}} < \frac{1}{3\times2^{n-1}}$ for $n \geq 4$,

so that $ln3 < 1 + \frac{1}{3\times2^2} + \frac{1}{3\times2^4} + \frac{1}{3\times2^6} + \cdots$

$= 1 + \frac{1}{3\times2^2} \cdot \frac{1}{1-\frac{1}{4}}$

$= 1 + \frac{1}{9} = \frac{10}{9}$ (but $\frac{10}{9} > \frac{11}{10}$)

Consider instead:

$ln3 < 1 + \frac{1}{12} + \frac{1}{5\times2^4} + \frac{1}{5\times2^6} + \cdots$

$= 1 + \frac{1}{12} + \frac{1}{5\times2^4} \cdot \frac{1}{1-\frac{1}{4}}$

$= \frac{13}{12} + \frac{1}{60} = \frac{65}{60} < \frac{66}{60} = \frac{11}{10}$

Thus $\frac{13}{12} < ln3 < \frac{11}{10}$, as required.

(v) [As $y = lnx$ is an increasing function,

we can compare $\ln(3^{17})$ and $\ln(4^{13})$ instead.]

Consider $\ln(3^{17}) - \ln(4^{13}) = 17ln3 - 13\ln(2^2)$

$= 17ln3 - 26ln2$

$> 17\left(\frac{13}{12}\right) - 26(\frac{17}{24})$, from (ii) & (iv)

$= \frac{17}{12}(13 - 13) = 0$

So $\ln(3^{17}) > \ln(4^{13})$,

and hence, as $y = lnx$ is an increasing function, $3^{17} > 4^{13}$