# 2021 MAT - Multiple Choice (7 pages; 9/9/22)

## Q1/A

The area is made up of 12 isosceles triangles with equal sides of length 1, and an angle of $30^\circ$ between these sides.

Hence area of dodecagon is $(12)\frac{1}{2}(1)(1)sin30°$

$= 6\left(\frac{1}{2}\right) = 3$ square units

**So the answer is (e).**

## Q1/B

$\int_0^a \sqrt{x} + x^2\, dx = 5 \Leftrightarrow \left[\frac{1}{\left(\frac{3}{2}\right)}x^{\frac{3}{2}} + \frac{1}{3}x^3\right]\begin{matrix} a \\ 0 \end{matrix} = 5$

$\Leftrightarrow \frac{2}{3}a^{\frac{3}{2}} + \frac{1}{3}a^3 = 5$

[Note that $a^3$ can be made the subject of all but one of the multiple choice options.]

Let $b = a^{\frac{3}{2}}$, so that $2b + b^2 = 15$, or $b^2 + 2b - 15 = 0$

$\Leftrightarrow (b + 5)(b - 3) = 0$

$\Leftrightarrow a^{\frac{3}{2}} = -5\ or\ 3$

And $a^{\frac{3}{2}} = 3$ when $a = 3^{\frac{2}{3}}$

**So the answer is (c).**

[To see if $a^{\frac{3}{2}} = -5$ has a solution:

$a^{\frac{3}{2}} = -5 \Rightarrow a^3 = 25 \Rightarrow a = +\sqrt[3]{25}$

But the graph of $y = a^x$ for positive $a$ always lies above the $x$ axis (If $a < 1$, then $y = (\frac{1}{b})^x = b^{-x}$, where $b > 1$), and so $a^{\frac{3}{2}} = -5$ is not possible.)]

**Q1/C**

[It is tempting to say that there can be no connection between $p - a$ & $q - b$, and so options (b)-(e) cannot be correct.]

The tangent to $y = e^x$ passing through $(p, e^p)$ has equation $\frac{y-e^p}{x-p} = e^p$, and crosses the $x$-axis at $(a, 0)$, where $\frac{0-e^p}{a-p} = e^p$,

so that $a - p = -1$

Thus $p - a = q - b = 1$

**So the answer is (c).**

**Q1/D**

[The graph of $y = 1 - e^x$ can be obtained by reflecting $y = e^x$ in the $x$-axis, and translating up by 1.]

The graph of $y = 1 - e^x$ passes through the Origin.

The graphs of $y = e^x$ and $y = 1 - e^x$ intersect when $e^x = 1 - e^x$

$\Leftrightarrow 2e^x = 1 \Leftrightarrow e^x = \frac{1}{2} \Leftrightarrow x = ln\left(\frac{1}{2}\right) = -ln2$

The required area is

$\int_{-ln2}^{0} e^x \ dx - \int_{-ln2}^{0} (1 - e^x) \ dx = \int_{-ln2}^{0} 2e^x - 1 \ dx$

$= [2e^x - x]_{-ln2}^{0} = 2 - (1 + ln2) = 1 - ln2$

**So the answer is (b).**

## Q1/E

Let the number of $\binom{1}{1}$ $s$ be $x$, so that the number of $\binom{3}{2}$ $s$ is

$6-x$.

Then we require $x+3(6-x)=10$ and $x+2(6-x)=8$;

ie $-2x=-8$, so that $x=4$;

and $-x=-4$, so that $x=4$ again

$P(x=4)=\binom{6}{4}(\frac{1}{2})^4(\frac{1}{2})^2=\binom{6}{2}.\frac{1}{2^6}=\frac{6(5)}{2^7}=\frac{15}{64}$

**So the answer is (c).**

## Q1/F

[From a sketch of $y=x^3-3x=x(x^2-3)$, it appears as though there will be 3 points on the curve for which the tangent passes through (2,0), but just to check:]

The gradient of the tangent is $3x^2-3$, with $x=a$, and so the equation of the tangent is $\frac{y-(a^3-3a)}{x-a}=3a^2-3$

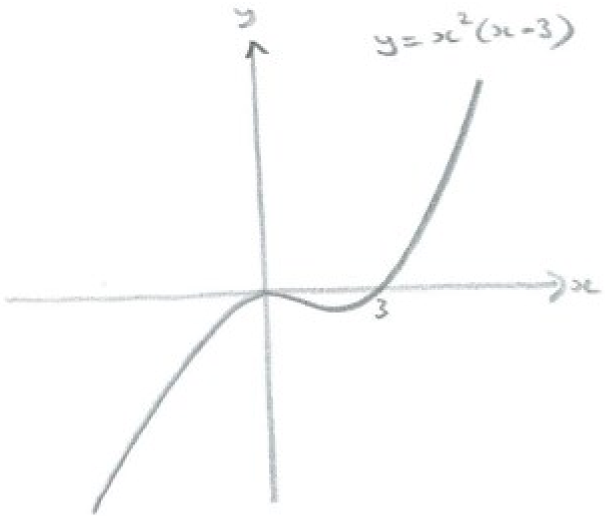
In order for the tangent to pass through (2,0), we require:

$\frac{0-(a^3-3a)}{2-a}=3a^2-3$,

giving $-a^3+3a=6a^2-6-3a^3+3a$,

so that $2a^3-6a^2+6=0$, or $a^3-3a^2+3=0$ (*)

Now the curve $y=x^3-3x^2=x^2(x-3)$ has a repeated root at $x=0$ and a root at $x=3$ (see diagram).




The number of real roots of (*) will depend on the $y$ coordinate of the minimum point when the curve is shifted up by 3.

To find the minimum point of $y = x^3 - 3x^2 + 3$:

$\frac{dy}{dx} = 0 \Rightarrow 3x^2 - 6x = 0 \Rightarrow x(x-2) = 0$

$x = 0$ corresponds to the maximum and $x = 2$ corresponds to the minimum

When $x = 2, y = 8 - 12 + 3 = -1$.

So the minimum point lies below the $x$-axis, and there are 3 real roots of (*).

**So the answer is (d).**

## Q1/G

[We can try to pair off the terms in some way. This suggests the use of $sin\theta = cos(90 - \theta)$.]

Now $sin^2\theta = 1 - cos^2\theta = 1 - sin^2(90° - \theta)$.

So $sin^2 1° + sin^2 89° = 1$

and so on, until $sin^2 44° + sin^2 46° = 1$

Thus, the required sum is:

$44 + sin^2 45° + sin^2 90° = 44 + \frac{1}{2} + 1 = 45.5$

[Reassuringly, not 45 – which is probably what many candidates will guess at.]

**So the answer is (d).**

**Q1/H**

The graph will cross the $x$-axis when

$9 - 8sinx - 6cos^2 x = 1$

ie when $9 - 8sinx - 6(1 - sin^2 x) = 1$

or $6sin^2 x - 8sinx + 2 = 0$

or $3sin^2 x - 4sinx + 1 = 0,$

so that $(3sinx - 1)(sinx - 1) = 0,$

and hence $sinx = \frac{1}{3}\ or\ 1$

**So the answer is (a).**

**Q1/I**

If $a_n$ is one more than the product of all previous terms, then $a_3 = 43$, and only (b) or (e) are consistent with this.

We then see that (b) is equivalent to this definition ($a_{n-1} - 1$ is the product of all terms up to and including $a_{n-2}$).

**So the answer is (b).**

**Q1/J**

The 4 sides are AB, BC, CD and DA, and the squares of these lengths are:

$(b-a)^2+(c-b)^2$

$(c-b)^2+(d-c)^2$

$(d-c)^2+(a-d)^2$

& $(a-d)^2+(b-a)^2$

So the 4 sides are the same length when

$(b-a)^2+(c-b)^2=(c-b)^2+(d-c)^2$

$=(d-c)^2+(a-d)^2=(a-d)^2+(b-a)^2$

which is equivalent to:

$(b-a)^2=(d-c)^2$ , $(c-b)^2=(a-d)^2$ and

$(d-c)^2=(b-a)^2$

ie just $(b-a)^2=(d-c)^2$ & $(c-b)^2=(a-d)^2$

There are 4 possibilities:

P: $b-a=d-c$ & $c-b=a-d$

$Q$: $b-a=d-c$ & $c-b=d-a$

$R$: $b-a=c-d$ & $c-b=a-d$

$S$: $b-a=c-d$ & $c-b=d-a$

$P\Rightarrow b-a=d-[a-d+b]\Rightarrow 2b=2d$, which isn't possible, as $a,b,c$ & $d$ are supposed to be distinct

$Q \Rightarrow b - a = d - [d - a + b] \Rightarrow 2b = 2a$, which also isn't possible

$R \Rightarrow b - a = [a - d + b] - d \Rightarrow 2d = 2a$, which also isn't possible

$S \Rightarrow b - a = [d - a + b] - d \Rightarrow 2d = 2a$, which IS possible

Both $b - a = c - d$ & $c - b = d - a$ can be written as

$a - b + c - d = 0$, so that if the 4 sides are the same length, then

$a - b + c - d = 0$

And if $a - b + c - d = 0$, then

$(b - a)^2 = (d - c)^2$ & $(c - b)^2 = (a - d)^2$

and the 4 sides will then be the same length.

**So the answer is (d).**