2020 MAT – Q6 (3 pages; 10/10/22)

Solution

(i) $g(1,k) = k$ [The single ball could be given to k people.]

(ii) $g(n,1) = 1$ [The balls are all given to the single person.]

(iii) The $g(n,k)$ ways can be classified as follows:

(a) The 1st child receives at least one ball. Without loss of generality, it can be given the 1st ball. Number of ways in which the remaining $n-1$ balls can be distributed to k children (including the 1st child) is $g(n-1,k)$.

(b) The 1st child receives no balls. Number of ways in which the n balls can be distributed to the other $k-1$ children is $g(n,k-1)$.

So $g(n,k) = g(n-1,k) + g(n,k-1)$.

(iv) Hence $g(7,5) = g(6,5) + g(7,4)$

$= [g(5,5) + g(6,4)] + [g(6,4) + g(7,3)]$

$= [g(4,5) + g(5,4)] + 2[g(5,4) + g(6,3)] + [g(6,3) + g(7,2)]$

$= g(4,5) + 3g(5,4) + 3g(6,3) + g(7,2)$

$= [g(3,5) + g(4,4)] + 3[g(4,4) + g(5,3)] + 3[g(5,3) + g(6,2)]$

$+[g(6,2) + g(7,1)]$

$= g(3,5) + 4g(4,4) + 6g(5,3) + 4g(6,2) + 1$

$= [g(2,5) + g(3,4)] + 4[g(3,4) + g(4,3)] + 6[g(4,3) + g(5,2)]$

$+4[g(5,2)+g(6,1)]+1$

$=g(2,5)+5g(3,4)+10g(4,3)+10g(5,2)+4+1$

$=[g(1,5)+g(2,4)]+5[g(2,4)+g(3,3)]+10[g(3,3)+$
$g(4,2)]+10[g(4,2)+g(5,1)]+5$

$=5+6g(2,4)+15g(3,3)+20g(4,2)+10+5$

$=6[g(1,4)+g(2,3)]+15[g(2,3)+g(3,2)]+20[g(3,2)+$
$g(4,1)]+20$

$=24+21g(2,3)+35g(3,2)+20+20$

$=21[g(1,3)+g(2,2)]+35[g(2,2)+g(3,1)]+64$

$=63+56(3)+35+64=330$

[Note: The official sol'ns employs the following table, which cuts down on the amount of working. $g(7,5)$ can in fact be built up by starting with $g(2,2)$:

$g(3,2)=g(2,2)+g(3,1)=3+1$ etc]

n	$g(n,1)$	$g(n,2)$	$g(n,3)$	$g(n,4)$	$g(n,5)$
1	1	2	3	4	5
2	1	3	6	10	15
3	1	4	10	20	35
4	1	5	15	35	70
5	1	6	21	56	126
6	1	7	28	84	210
7	1	8	36	120	330

Alternative (quicker) method (allowed by Official Sol'ns)

For $g(7,5)$, suppose for example that the 1st child receives 2 balls, the 2nd child receives 1 ball, the 3rd child receives 0 balls, the 4th child receives 1 ball, and the 5th child receive 3 balls. This can be denoted by $XX|X||X|XXX$ (where an | indicates that we are moving on to the next child).

Then $g(7,5)$ is the number of ways of choosing 4 positions for the |s out of the available $7+4$ positions.

So $g(7,5) = \binom{11}{4} = \frac{11(10)(9)(8)}{4!} = \frac{11(10)(9)(8)}{24} = 11(5)(3)(2) =$ 330.

[In general, $g(n,k) = \binom{n+(k-1)}{k-1}$]

(v) First of all, a ball can be given to each of the k children (assuming $n \geq k$). The number of ways of handing out the balls then equals $g(n-k,k)$.

Thus $h(7,5) = g(2,5) = g(1,5) + g(2,4)$

$= 5 + g(1,4) + g(2,3)$

$= 5 + 4 + g(1,3) + g(2,2)$

$= 9 + 3 + 3 = 15$