2020 MAT – Q4 (4 pages; 26/10/21)

(There is a typo. in the 2nd line of the question. It should read

"A function is said to be odd if $f(-x) = -f(x)$ [rather than

$f(-x) = -f(-x)$])

(i) (a) An even function has reflective symmetry about the y-axis.

An odd function has rotational symmetry of order 2 about the Origin.

(b) Consider the gradient of an even function when $x = a$, $f'(a)$.

Then, by the symmetry in the y-axis, the gradient at $x = -a$ will be $-f'(a)$

Thus $f'(-a) = -f'(a)$, so that the derivative of an even function is an odd function.

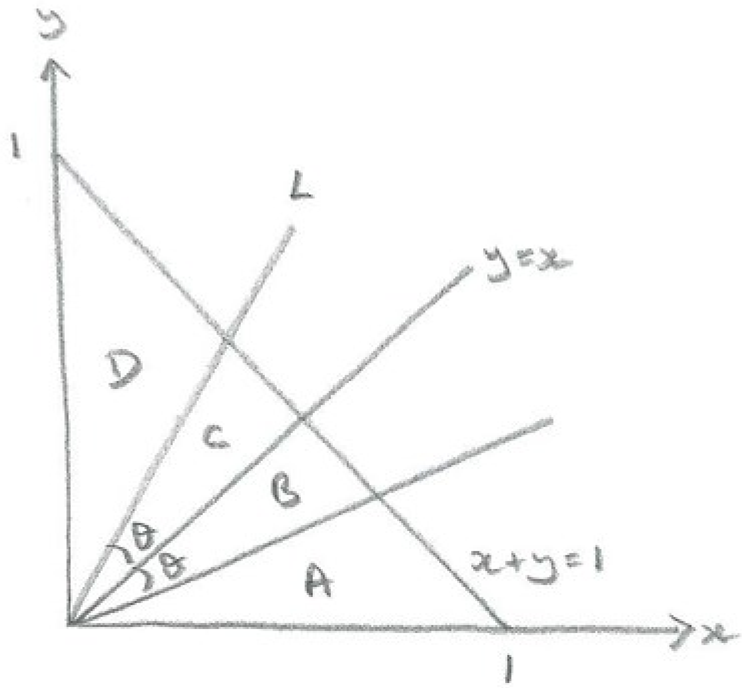
Now consider the gradient of an odd function when $x = a$, $g'(a)$.

A rotation of 180° is equivalent to a reflection in the y-axis, followed by a reflection in the x-axis. The gradient changes its sign (but not its magnitude) with each reflection, and so $g'(-a) = -[-g'(a)] = g'(a)$.

Thus the derivative of an odd function is an even function.

(ii) Referring to the diagram below,

$A(-\theta) = A$ and $A(\theta) = A + B + C$



And, by symmetry, $A = D$.

Then $A(\theta) + A(-\theta) = (A + B + C) + A = A + B + C + D = \frac{1}{2}$,
as required.

(b) Let L intersect the line $x + y = 1$ at the point $P(a, b)$.

Then $A(\theta) = \frac{1}{2}(1)b$

Also, $\frac{b}{a} = \tan\left(\theta + \frac{\pi}{4}\right) = \frac{tan\theta+1}{1-tan\theta}$ and $a + b = 1$,

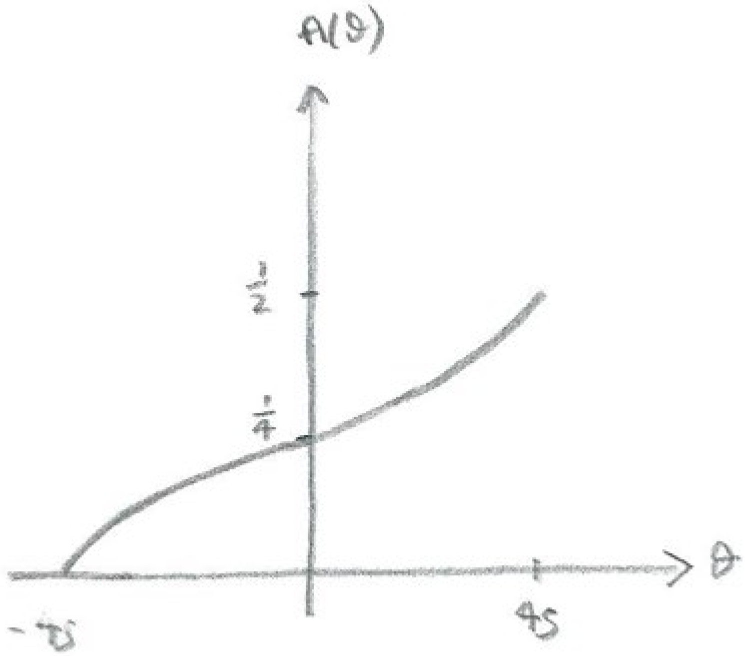
so that $\frac{b(1-tan\theta)}{1+tan\theta} + b = 1$,

and hence $b(1 - tan\theta) + b(1 + tan\theta) = 1 + tan\theta$,

so that $b = \frac{1}{2}(1 + tan\theta)$ and therefore $A(\theta) = \frac{1}{4}(1 + tan\theta)$

[Check: $A(0) = \frac{1}{4}$ and $A\left(\frac{\pi}{4}\right) = \frac{1}{2}$]

(c)



(d) From (ii)(c), $A(\theta)$ has rotational symmetry of order 2 about the point $\left(0, \frac{1}{4}\right)$.

This can be deduced from (ii)(a) as follows:

Let $B(\theta) = A(\theta) - \frac{1}{4}$

Result to prove: $B(-\theta) = -B(\theta)$ or $B(-\theta) + B(\theta) = 0$

Now $B(-\theta) + B(\theta) = \left(A(-\theta) - \frac{1}{4}\right) + (A(\theta) - \frac{1}{4})$

$= A(-\theta) + A(\theta) - \frac{1}{2}$

$= 0$, from (ii)(a)

(e) The rate of increase of the area $A(\theta)$ falls as θ increases from 0 to 45 (as the incremental area reduces) until $\theta = 0$, when it increases again. Thus the rate of increase of the area ($\frac{dA}{d\theta}$) has a minimum at $\theta = 0$, and so $\frac{d^2A}{d\theta^2} = \frac{d}{d\theta}\left(\frac{dA}{d\theta}\right) = 0$ at this point (a point of inflexion).