2020 MAT - Multiple Choice (6 pages; 26/10/22)

Q1/A

The corner opposite $(1, 5)$ will be $(3 + [3 - 1], 4 + [4 - 5])$;

ie $(5, 3)$

Let one of the other corners be at (a, b).

The distance of the corner $(1, 5)$ from the centre $(3, 4)$ is $\sqrt{2^2 + 1^2} = \sqrt{5}$, and hence each side of the square is of length $\sqrt{2}.\sqrt{5} = \sqrt{10}$

So the distance of (a, b) from each of $(1, 5)$ and $(5, 3)$ is $\sqrt{10}$

ie $(a - 1)^2 + (b - 5)^2 = 10$ (1)

and $(a - 5)^2 + (b - 3)^2 = 10$ (2)

Subtracting (2) from (1):

$-2a - 10b + 26 - (-10a) - 34 - (-6b) = 0$;

ie $8a - 4b = 8; b = 2a - 2$

[only (d) satisfies this]

Then substituting into (1):

$(a - 1)^2 + (2a - 7)^2 = 10$,

so that $5a^2 - 30a + 40 = 0$,

and $a^2 - 6a + 8 = 0$,

giving $(a - 4)(a - 2) = 0$,

so that the other two corners are $(4, 6)$ and $(2, 2)$

So the answer is (d).

Q1/B

$\int_0^1 (e^x - x)(e^x + x)\,dx = \int_0^1 e^{2x} - x^2\,dx$

$= \left[\frac{1}{2}e^{2x} - \frac{1}{3}x^3\right]\begin{matrix}1\\0\end{matrix}$

$= \left(\frac{1}{2}e^2 - \frac{1}{3}\right) - (\frac{1}{2} - 0)$

$= \frac{1}{6}(3e^2 - 5)$

So the answer is (d).

Q1/C

$1 - 4 + 9 - 16 + \cdots + 99^2 - 100^2$

$= \sum_{r=1}^{50}\{(2r-1)^2 - (2r)^2\}$

$= \sum_{r=1}^{50}(-4r + 1)$

$= -4\left(\frac{1}{2}\right)(50)(51) + 50$

$= 50(-102 + 1)$

$= -50(101)$

$= -5050$

So the answer is (e).

Q1/D

$3cos^2x + 2sinx + 1 = 3(1 - sin^2x) + 2sinx + 1$

$= -3sin^2x + 2sinx + 4$

$= -3\left(sinx - \frac{1}{3}\right)^2 + \frac{1}{3} + 4$

When $sinx = \frac{1}{3}$, this has its largest value of $\frac{13}{3}$

So the answer is (b).

Q1/E

$y = x^2 \Rightarrow \frac{dy}{dx} = 2x,$

So the gradient of the tangent at (a, a^2) is $2a$,

And the equation of the tangent is $\frac{y-a^2}{x-a} = 2a$

or $y = 2ax - a^2$,

and the tangent therefore crosses the x-axis at $(\frac{a}{2}, 0)$

Then the required area $= \int_0^a x^2\, dx - \int_{\frac{a}{2}}^a 2ax - a^2\, dx$

$= \left[\frac{1}{3}x^3\right]_0^a - [ax^2 - a^2x]_{\frac{a}{2}}^a$

$= \frac{1}{3}a^3 - (0 - \frac{1}{4}a^3 + \frac{1}{2}a^3)$

$= \frac{1}{12}a^3(4 + 3 - 6)$

$= \frac{1}{12}a^3$

So the answer is (c).

Q1/F

$log_{10}(10 \times 9 \times 8 \times \ldots \times 2 \times 1)$

$= log_{10}(10) + log_{10}(9) + log_{10}(8) + \cdots + log_{10}(2)$

$= 1 + 2log_{10}(3) + 3log_{10}(2) + log_{10}(7) + log_{10}(6)$

$+log_{10}(5) + 2log_{10}(2) + log_{10}(3) + log_{10}(2)$

$= 1 + 3log_{10}(3) + 6log_{10}(2) + log_{10}(7) + log_{10}(6)$

$+log_{10}(5)$

$= 1 + 3log_{10}(3) + 5log_{10}(2) + log_{10}(7) + log_{10}(6)$

$+log_{10}(10)$

$= 2 + 3log_{10}(3) + 5log_{10}(2) + log_{10}(7) + log_{10}(6)$

$= 2 + 3(log_{10}(3) + log_{10}(2)) + 2log_{10}(2) + log_{10}(7)$

$+log_{10}(6)$

$= 2 + 4log_{10}(6) + 2log_{10}(2) + log_{10}(7)$

So the answer is (c).

Q1/G

$y = x^3 + ax^2 + bx + c \Rightarrow \frac{dy}{dx} = 3x^2 + 2ax + b$

At the turning points, $\frac{dy}{dx} = 0$,

So that $3 + 2a + b = 0$ (1) and $27 + 6a + b = 0$ (2)

Subtracting (1) from (2), $24 + 4a = 0; a = -6$

Then, from (1), $b = 9$

Hence, when $x = 1, y = 1 - 6 + 9 + c$, so that $c + 4 = 2$,

and hence $c = -2$

And when $x = 3, y = 27 - 54 + 27 - 2 = -2$

Hence $d = -2$

So the answer is (b).

Q1/H

[Noting that cubic graphs have rotational symmetry (about their point of inflexion), it would seem that (b), (d) & (e) are cubics, whilst (c) is a quadratic, and (a) is a quartic.]

By considering the gradient of (a) at various points, (e) can be seen to be the derivative of (a).

Similarly, (c) is the derivative of (b).

So the answer is (d).

Q1/I

The sum to infinity of the GP $\frac{1}{tanx}+\frac{1}{tan^2x}+\frac{1}{tan^3x}+\cdots$

is $\frac{1}{tanx}\cdot\frac{1}{1-\frac{1}{tanx}}=\frac{1}{tanx-1}$, provided that $\left|\frac{1}{tanx}\right|<1$; ie $|tanx|>1$

Then $\frac{1}{tanx-1}=tanx\Rightarrow 1=tan^2x-tanx$

$\Rightarrow tan^2x-tanx-1=0$

$\Rightarrow tanx=\frac{1\pm\sqrt{5}}{2}$

But $\left|\frac{1-\sqrt{5}}{2}\right|<1$, so there is only one value of $tanx$, and hence only one value of x within the given range.

So the answer is (b).

Q1/J

Let $C(r)=A(r)+B(r)$

$C(0)=0+4$, which eliminates (c)

$C(r) = 0 \Rightarrow A(r) = B(r) = 0$, which is impossible, so (b) & (e) can be eliminated

For $r > \sqrt{2}, C(r) = (\pi r^2 - 4) + 0$

so that $C'(r) = 2\pi r$, and the gradient therefore increases with r.

This eliminates (a), as its gradient is reducing for $r > \sqrt{2}$.

This only leaves (d).

So the answer is (d).