2019 MAT – Q5 (2 pages; 27/10/21)

(i) Each subset must contain at least 2 elements. So if there are k subsets then there must be at least $2k$ elements in total. If $k > \frac{n}{2}$ then $2k > n$, so that the n elements available are not sufficient to produce the required total of at least $2k$.

(ii) If $n = 1$, then $f(n,1) = 0$, as subsets are required to contain at least 2 elements.

If $n > 1$, then $f(n,1) = 1$, as there is only one possible partition:$\{1,2,3, \dots , n\}$.

(iii) There will be two types of partition:

(a) Ones where $n + 1$ appears in a subset of size 2, and (b) Ones where $n + 1$ only appears in a subset of size greater than 2.

For (a), $n + 1$ could be paired with any of the numbers $1,2,3, \dots , n$, and in each case there will be $f(n - 1, k - 1)$ ways of partioning the remaining $n - 1$ numbers into $k - 1$ subsets (bearing in mind that the pair involving $n + 1$ makes up the kth subset). However, this only holds if $k \geq 2$ (as, with $k = 1$, $f(n - 1,0)$ is not defined).

ie there are $nf(n - 1, k - 1)$ such partitions

For (b), $n + 1$ can be thought of as being tacked onto one of $f(n, k)$ partitions occuring when there are just n numbers. In each case, there are k possible subsets for $n + 1$ to be tacked onto.

ie there are $kf(n, k)$ such partitions

Hence $f(n + 1, k) = nf(n - 1, k - 1) + kf(n, k)$, and this is valid

for $k \geq 2$ and $k \leq n$, and so for $2 \leq k < n$ as well.

[It isn't clear why the range $2 \leq k \leq n$ hasn't been chosen.]

(iv) From (iii), $f(7,3) = 6f(5,2) + 3f(6,3)$

$= 6[4f(3,1) + 2f(4,2)] + 3[5f(4,2) + 3f(5,3)]$

$= 24 + 27f(4,2) + 9f(5,3)$

But $f(n,k) = 0$ when $k > \frac{n}{2}$, from (i), and so

$f(7,3) = 24 + 27f(4,2)$

$= 24 + 27[3f(2,1) + 2f(3,2)]$

$= 105$, as $f(2,1) = 1$ [from (i)], and $f(3,2) = 0$, as $k > \frac{n}{2}$

(v) Each subset must contain at least 2 elements, so if there are only $2n$ elements in total, each of the n subsets must contain exactly 2 elements.

There are $\binom{2n}{2}$ ways of selecting the 1st subset; then, for each of these, there are $\binom{2n-2}{2}$ ways of selecting the 2nd subset, but noting that AB counts as a different choice from BA (where A & B are subsets), so that there is over-counting.

So the number of ways of selecting the n subsets (with over-counting) is $\binom{2n}{2}\binom{2n-2}{2}\ldots\binom{2}{2}$

$= \frac{2n(2n-1)(2n-2)(2n-3)\ldots 2(1)}{(2!)^n} = \frac{(2n)!}{2^n}$

As there are $n!$ ways of arranging the subsets, the over-counting can be removed by dividing by $n!$, to give

$f(2n,n) = \frac{(2n)!}{n!2^n}$