2019 MAT - Q3 (3 pages; 6/11/20)

(i) **1st part**

$\int_0^c x(c-x)dx = \left[\frac{1}{2}cx^2 - \frac{1}{3}x^3\right]_0^c = \frac{1}{2}c^3 - \frac{1}{3}c^3 = \frac{1}{6}c^3$

2nd part

[The area of S is $\int_a^b (x-a)(b-x)dx$

Let $y = x - a$. Then $S = \int_0^{b-a} y(b-y-a)dy$

Let $c = b - a$. Then $S = \int_0^c y(c-y)dy = \frac{1}{6}c^3 = \frac{1}{6}(b-a)^3$

However the question setter is presumably interested in the idea of translating S a distance a to the left, rather than making a substitution (although this is the algebraic equivalent).]

Consider the region S' under the curve obtained by translating the given curve by a distance a to the left (so that the areas of S' & S are equal).

Then the function for the new curve is obtained from that of the original curve by replacing x with $x + a$, so that $(x-a)(b-x)$ becomes $x(b-[x+a]) = x(c-x)$, if we write $c = b - a$, and the limits of integration (when determining the area under the curve) change from a & b to 0 & $b - a = c$

Thus the area of S equals $\int_0^c x(c-x)dx = \frac{1}{6}c^3 = \frac{1}{6}(b-a)^3$

[Arguably, some calculation is needed, in order to establish that $b-(x+a) = c - x$]

(ii) Let $f(x) = (x-a)(b-x)$

Then $f'(x) = (b-x) + (x-a)(-1) = -2x + b + a$

and $f'(x) = m \Rightarrow x = \frac{b+a-m}{2}$ (1)

Also, $f(x) = mx$,

so that $(x-a)(b-x) = mx$,

and so from (1), $\left(\frac{b+a-m}{2} - a\right)\left(b - \frac{b+a-m}{2}\right) = m(\frac{b+a-m}{2})$

$\Rightarrow (b+a-m-2a)(2b-b-a+m) = 2m(b+a-m)$

$\Rightarrow (b-a-m)(b-a+m) = 2m(b+a-m)$

$\Rightarrow (b-a)^2 - m^2 = 2(b+a)m - 2m^2$

$\Rightarrow m^2 - 2(b+a)m + (b-a)^2 = 0$

$\Rightarrow m = \frac{2(b+a) \pm \sqrt{4(b+a)^2 - 4(b-a)^2}}{2}$

$= b + a \pm \sqrt{4ab}$ (2)

And $(\sqrt{b} - \sqrt{a})^2 = b + a - 2\sqrt{ab}$, which is the smaller of the two sol'ns in (2).

To show that $m = b + a + 2\sqrt{ab}$ isn't possible:

$x = \frac{b+a-m}{2}$, from (1),

and so the larger sol'n $\Rightarrow x = \frac{-2\sqrt{ab}}{2} < 0$, which can be rejected.

[With hindsight, this complication could have been avoided if we had solved a quadratic in x, and then eliminated the negative root. Also, the method of using the discriminant, in the official sol'ns, is quicker - though the choice of root still has to be justified.]

(iii) **1st part**

From (i), $S = \frac{1}{6}(\beta^2 - 1)^3$,

so $R = S \Leftrightarrow \frac{(2\beta+1)(\beta-1)^2}{6} = \frac{1}{6}(\beta^2 - 1)^3$

$\Leftrightarrow (\beta - 1)^2\{2\beta + 1 - (\beta + 1)^2(\beta^2 - 1)\} = 0$

$\Leftrightarrow (\beta - 1)^2\{2\beta + 1 - (\beta^2 + 2\beta + 1)(\beta^2 - 1)\} = 0$

$\Leftrightarrow (\beta - 1)^2\{2\beta + 1 - (\beta^4 - \beta^2 + 2\beta^3 - 2\beta + \beta^2 - 1)\} = 0$

$\Leftrightarrow (\beta - 1)^2\{-\beta^4 - 2\beta^3 + 4\beta + 2\} = 0$

$\Leftrightarrow (\beta - 1)^2\{\beta^4 + 2\beta^3 - 4\beta - 2\} = 0$, as required.

2nd part

Let $f(\beta) = \beta^4 + 2\beta^3 - 4\beta - 2$

Then $f(1) = 1 + 2 - 4 - 2 = -3$

Also $f(\beta) \to \infty$ as $\beta \to \infty$, so that $f(\beta) > 0$ for sufficiently large β.

Hence there is a change of sign, and therefore a root for some

$\beta > 1$ (as $f(\beta)$ is a continuous function).

3rd part

If $a(> 0) \neq 1$, we can change the scale on the x-axis, so that $x' = \frac{x}{a}$, and then find a b' that gives $S = R$. Then the required b equals

$b'a$.