2019 MAT - Q2 (2 pages; 5/11/20)

(i) $N = 1 + 2 + 3 + \cdots + k = \frac{1}{2}k(k+1)$

(ii) $p_k(1) = 2^k$

and also

$p_k(1) = a_0 + a_1 + \cdots + a_N \leq (N+1)a_{max}$ (as $a_i \leq a_{max}$ for all i)

So $2^k \leq (N+1)a_{max}$, and $a_{max} \geq \frac{2^k}{N+1}$, as required.

(iii) Once $k > i$, multiplication by $1 + x^k$ has no effect on a_i (multiplication by 1 leaves a_i unchanged, and multiplication by x^k only introduces coefficients of x^r for $r \geq k$)

(iv) $p_k(x^{-1}) = (1 + x^{-1})(1 + x^{-2}) \ldots (1 + x^{-k})$

$= x^{-(1+2+\cdots+k)}(x+1)(x^2+1) \ldots (x^k+1) = x^{-N}p_k(x)$

$= a_0x^{-N} + a_1x^{-N+1} + \cdots + a_Nx^0$

Also, $p_k(x^{-1}) = a_0 + a_1x^{-1} + \cdots + a_Nx^{-N}$

Then, equating coefficients of x^{-N+i}:

$a_i = a_{N-i}$ (valid for $0 \leq i \leq N$), as required.

(v) If N is odd (eg 21, as in the student's example), then there will be $N + 1$ coefficients, and (by the symmetry demonstrated) the first $\frac{N+1}{2}$ coefficients must feature all of the numbers $1, 2, \ldots, a_{max}$, if the student's guess is to be correct. We want to show that, for at least one value of k, $a_{max} > \frac{N+1}{2}$

Now, from (ii), $a_{max} \geq \frac{2^k}{N+1}$, so we need to show that

$\frac{2^k}{N+1} > \frac{N+1}{2}$, for some k;

ie that $2^{k+1} > (N+1)^2 = (\frac{1}{2}k(k+1)+1)^2$, from (i);

ie that $2^{k+3} > (k(k+1)+2)^2$

As the LHS is an exponential function of k, whilst the RHS is only a quartic in k, this will be true for a sufficiently high value of k.

If instead N is even, then the first $\frac{N}{2}$ coefficients must feature all of the numbers $1, 2, \ldots, a_{max}$, and we need to show that

$\frac{2^k}{N+1} > \frac{N}{2}$, for some k;

ie that $2^{k+1} > N(N+1)$, and the RHS will again be a quartic in k, and the same conclusion is reached.