2018 MAT – Q6 (4 pages; 13/11/23)

(i) $\frac{2}{3} = \frac{1}{2} + \frac{1}{6}$ ($\frac{1}{2}$ is the 1st reciprocal we can try, and $\frac{2}{3} - \frac{1}{2} = \frac{1}{6}$)

For $\frac{2}{5}$, $\frac{1}{3}$ is the 1st reciprocal we can try, and $\frac{2}{5} - \frac{1}{3} = \frac{1}{15}$,

So $\frac{2}{5} = \frac{1}{3} + \frac{1}{15}$

For $\frac{23}{40}$, $\frac{1}{2}$ is the 1st reciprocal we can try, and $\frac{23}{40} - \frac{1}{2} = \frac{3}{40}$

Then we can try to find a friendly form for $\frac{3}{40}$.

As $\frac{3}{40} = \frac{1}{(\frac{40}{3})} = \frac{1}{13+a}$, where $0 < a < 1$, $\frac{1}{14}$ is the 1st reciprocal we can try, and $\frac{3}{40} - \frac{1}{14} = \frac{2}{560} = \frac{1}{280}$

So $\frac{23}{40} = \frac{1}{2} + \frac{1}{14} + \frac{1}{280}$

(ii) m has to satisfy $\frac{1}{m} \leq q$, so that $m \geq \frac{1}{q} = \frac{b}{a}$, and m must be the smallest such number, so that $\frac{1}{m-1} > q$; ie $m + 1 < \frac{1}{q} = \frac{b}{a}$

Hence $\frac{b}{a} \leq m < \frac{b}{a} + 1$ (**)

And $\frac{a}{b} - \frac{1}{m} = \frac{c}{d}$, where c & d have no common factors (as well as a & b).

Result to prove: $c < a$ (*)

We can try to prove instead that, as $\frac{a}{b} - \frac{1}{m} = \frac{ma-b}{bm}$,

it is the case that $ma - b < a$ (from which (*) follows, since if $\frac{c}{d}$ is the simplest form of $\frac{ma-b}{bm}$, then $c \leq ma - b$)

This is equivalent to showing that $m < \frac{a+b}{a}$ (as $a > 0$);

Or $m < 1 + \frac{b}{a}$ (which is the case). Hence $c < a$.

(iii) Given $q = \frac{a}{b}$, m can be found as in (ii); ie m is the integer satisfying $\frac{b}{a} \leq m < \frac{b}{a} + 1$ (*)

Then $q - \frac{1}{m} = \frac{c}{d}$, and $q = \frac{1}{m} + \frac{c}{d}$, with $c < a$ (**)

The procedure is then repeated for $q_1 = \frac{c}{d}$, to produce

$q_1 - \frac{1}{m_1} = \frac{c_1}{d_1}$, with $c_1 < c$

However, we have to show that $m_1 > m$

[Referring to the examples in (i), it seems that m_1 is generally significantly bigger than m, suggesting that it may not be hard to prove that $m_1 > m$.]

Now $\frac{b}{a} \leq m < \frac{b}{a} + 1$ and $\frac{d}{c} \leq m_1 < \frac{d}{c} + 1$, where $\frac{c}{d} < \frac{a}{b}$,

so that $\frac{d}{c} > \frac{b}{a}$, and hence $m_1 \geq m$

Now suppose that $m_1 = m$.

Then we have $q - \frac{1}{m} - \frac{1}{m} = \frac{c_1}{d_1}$ (*)

Does this perhaps contradict the fact that $q - \frac{1}{m-1} < 0$ (as m is supposed to be the smallest integer such that $\frac{1}{m} \leq q$; ie such that

$q - \frac{1}{m} \geq 0$)?

Consider $\frac{2}{m} - \frac{1}{m-1}$ (with a view to showing that this will be positive; ie that $\frac{2}{m} > \frac{1}{m-1}$)

$\frac{2}{m} - \frac{1}{m-1} = \frac{2(m-1)-m}{m(m-1)} = \frac{m-2}{m(m-1)} > 0$, provided that $m > 2$.

Thus, when $m > 2$, $\frac{2}{m} > \frac{1}{m-1}$,

and then $q - \frac{1}{m-1} < 0 \Rightarrow q - \frac{2}{m} < 0$, so that (*) is not possible, and therefore $m_1 \neq m$.

Now, $m = 1$ is not possible, as this gives $q - \frac{1}{m} \geq 0$,

so that $q \geq 1$, but we are told that $q < 1$.

And if $m = 2$, then $q - \frac{2}{m} \geq 0 \Rightarrow q \geq 1$ also.

Thus, $m_1 > m$, as required.

The process is then repeated, to give

$q_2 - \frac{1}{m_2} = \frac{c_2}{d_2}$, with $c_2 < c_1 < c < a$,

and so on until we reach $c_n = 0$ (after a finite number of steps)

(noting that, at each stage, $\frac{1}{m_r} \leq q_r$, so that $q_r - \frac{1}{m_r} \geq 0$, and hence $c_r \geq 0$)

Thus we arrive at $q - \frac{1}{m} - \frac{1}{m_1} - \cdots - \frac{1}{m_n} = 0$,

and so $q = \frac{1}{m} + \frac{1}{m_1} + \cdots + \frac{1}{m_n}$, where the m_i are distinct.

(iv) $\frac{4}{13} = \frac{1}{\left(\frac{13}{4}\right)} = \frac{1}{3+a}$, where $0 < a < 1$,

so that we can write $\frac{4}{13} - \frac{1}{4} = \frac{c}{d}$ (with $\frac{13}{4} \leq 4 < \frac{13}{4} + 1$)

and $\frac{c}{d} = \frac{4}{13} - \frac{1}{4} = \frac{3}{52}$

Then $\frac{3}{52} = \frac{1}{(\frac{52}{3})} = \frac{1}{17+b}$, where $0 < b < 1$,

so that we can write $\frac{3}{52} - \frac{1}{18} = \frac{e}{f}$ (with $\frac{52}{3} \leq 18 < \frac{52}{3} + 1$)

and $\frac{e}{f} = \frac{3}{52} - \frac{1}{18} = \frac{2}{52(18)} = \frac{1}{468}$

Thus $\frac{4}{13} = \frac{1}{4} + \frac{1}{18} + \frac{1}{468}$

(v) Let R be any rational number.

Then we can write $R = (\sum_{n=1}^{N} \frac{1}{n}) + \frac{a}{b}$, where $0 \leq \frac{a}{b} < \frac{1}{N+1}$, for some N. (#)

We need to show that $\frac{a}{b}$ can be written in the friendly form

$\frac{1}{m} + \frac{1}{m_1} + \cdots$, such that $m > N$ (so that all the reciprocals in the expansion for R are distinct).

From (**) in (ii), $\frac{b}{a} \leq m < \frac{b}{a} + 1$,

so that $m \geq \frac{b}{a} > N + 1$, from (#),

and therefore $m > N + 1 > N$, as required.