2018 MAT paper - Q2 (3 pages; 18/10/23)

Solution

(i) $TS(x,y) = T(x+1,y) = (-y,x+1)$

and $ST(x,y) = S(-y,x) = (-y+1,x)$

Thus $TS(x,y) \neq ST(x,y)$

(ii) $T^2(x,y) = T(-y,x) = (-x,-y)$

$T^3(x,y) = T(-x,-y) = (y,-x)$

$T^4(x,y) = T(y,-x) = (x,y)$

Thus $T^n(x,y) = T(x,y)$ when n is a multiple of 4.

(iii) **1st Part**

$(x,y) \rightarrow (-x,-y)$ from T^2

Then $(-x,-y) \rightarrow (-x+1,-y)$ from S

And $(-x+1,-y) \rightarrow (x-1,y)$ from T^2

Thus $T^2ST^2(x,y) = (x-1,y)$

2nd Part

Justification for 5 being the minimum number of transformations needed:

At least one S is required, in order to produce the 1 in $(x-1,y)$.

Each application of T causes the x to switch from appearing in the 1st to appearing in the 2nd coordinate, or vice-versa. Hence, to obtain $(x-1,y)$ from (x,y), there must be an even number of Ts.

But each application of T that leaves the x in the 1st coordinate, also changes the sign of the x, so that the required number of Ts cannot be two (otherwise we would be left with a $-x$ in the 1st coordinate). But as no Ts is not possible (as no -1 would ever appear), the minimum number of Ts is 4, which together with the required S means that 5 is the minimum total number of transformations.

(iv) $(a, 0)$ can be obtained from $(0,0)$ by a or $-a$ applications of either S or U (depending on whether a is positive or negative).

To show that (a, b) can then be obtained from $(a, 0)$:

$T^3(x, y) = (y, -x)$, from (ii)

Then $ST^3(x, y) = (y + 1, -x)$

and $TST^3(x, y) = (x, y + 1)$

Similarly, $TUT^3(x, y) = (x, y - 1)$

Thus b or $-b$ applications of either S or U (depending on whether b is positive or negative) will convert $(a, 0)$ to (a, b).

And so (a, b) can be obtained from $(0,0)$.

(v) **1st Part**

C has equation $y = (x + 1)^2 + 1$

S represents a translation of 1 in the x direction

So, applying S to C produces $y = ([x - 1] + 1)^2 + 1 = x^2 + 1$

2nd Part

T represents a reflection in $y = x$, followed by a reflection in the y-axis.

The reflection in $y = x$ produces $x = (y + 1)^2 + 1$,

And then the reflection in the y-axis produces $-x = (y + 1)^2 + 1$

Or $x + (y + 1)^2 + 1 = 0$