2017 MAT – Q7 (2 pages; 14/10/22)

Solution

(i) $R(a+b) = R(b) + R(a)$

$R\big(R(a)\big) = a$

(ii) $S_k(a+b) = b + R(a)$

$S_k\big(S_k(a+b)\big) = S_k(b + R(a)) = R(a) + R(b)$

(iii) $S_5(a) = (6,7,8,5,4,3,2,1)$

$S_5\big(S_5(a)\big) = (3,2,1,4,5,8,7,6) = (3,2,1) + (4,5) + (8,7,6)$

$S_5(S_5\big(S_5(a)\big)) = (8,7,6,5,4,1,2,3)$

$S_5(S_5(S_5\big(S_5(a)\big))) = (1,2,3,4,5,6,7,8) = a$

(iv) Let $a = b + c$, where b has length k and c has length $n-k$

Then $S_k(a) = c + R(b)$

Now c has length $n - k \leq \frac{n}{2} \leq k$.

Let $R(b) = d + e$, where d has length $k - (n-k) = 2k - n$

and e has length $k - (2k - n) = n - k$

Then $S_k(S_k(a)) = e + R(c + d) = e + R(d) + R(c)$

Now $e + R(d)$ has length $(n-k) + (2k-n) = k$

So $S_k(S_k(S_k(a))) = R(c) + R(e + R(d))$

$= R(c) + R\big(R(d)\big) + R(e) = R(c) + d + R(e)$

As e, and hence $R(e)$, has length $n-k$, $R(c)+d$ has length k (as $S_k(S_k(S_k(a)))$ has the same length as a).

So $S_k(S_k(S_k(S_k(a)))) = R(e) + R(R(c)+d)$

$= R(e) + R(d) + R\big(R(c)\big) = R(e) + R(d) + c$

As $R(b) = d + e$ (as defined earlier),

$b = R\big(R(b)\big) = R(d+e) = R(e) + R(d)$

So $S_k(S_k(S_k(S_k(a)))) = b + c = a$, as required.

(v) Consider $a = 1,2,3,4,5$ with $k = 2$

[Probably a better bet than $a = 1,2,3$ with $k = 1$, in order not to have the original order after 4 applications.]

Then $S_2(a) = 3,4,5,2,1$

$S_2(S_2(a)) = 5,2,1,4,3$

$S_2(S_2(S_2(a))) = 1,4,3,2,5$

$S_2(S_2(S_2(S_2(a)))) = 3,2,5,4,1$; ie not the original order

Performing S_2 5 times gives: 5,4,1,2,3

Performing S_2 6 times gives 1,2,3,4,5 ; so S_2 has to be performed 6 times.