2017 MAT – Q6 (4 pages; 12/10/22)

Solution

(i) The other 4 orderings can be written out as follows, where a X indicates an unsafe packing order:

	w_i	s_i	weight above
B	4	4	
C	12	9	4 Y
A	5	6	16 X

	w_i	s_i	weight above
C	12	9	
B	4	4	12 X
A	5	6	16 X

	w_i	s_i	weight above
A	5	6	
C	12	9	5 Y
B	4	4	17 X

	w_i	s_i	weight above
C	12	9	
A	5	6	12 X
B	4	4	17 X

So none of the 4 are safe.

(ii) [It is slightly unclear whether we are supposed to be dealing with the general case here, or just the Apples, Bread & Carrots case. However, there is only one safe order for the Apples, Bread & Carrots example, and this occurs with the weights in the suggested order (ie heaviest at the bottom etc) – thereby not providing a counter-example).

The following provides a counter-example:

	w_i	s_i	weight above
X	4	4	
Y	5	11	
Z	6	8	$4 + 5 > 8\ X$

	w_i	s_i	weight above
X	4	4	
Z	6	8	$4 < 8\ Y$
Y	5	11	$4 + 6 < 11\ Y$

(iii) The following provides a counter-example:

	w_i	s_i	weight above
X	2	4	
Y	5	5	
Z	3	6	$2 + 5 > 6\ X$

	w_i	s_i	weight above
X	2	4	
Z	3	6	$2 < 6\ Y$
Y	5	5	$2 + 3 = 5\ Y$

(iv)

Before

W	
w_j	s_j
w_i	s_i

After

W	
w_i	s_i
w_j	s_j

where W is the weight above the jth item initially

The only item that could be adversely affected by the swap is item j. After the swap the order will still be safe if $W + w_i \leq s_j$ or $s_j - w_i - W \geq 0$

We are told that $w_j - s_i \geq w_i - s_j$ (1)

and also, because the initial order was safe from the point of view of item i), $W + w_j \leq s_i$ (2)

From (1), $s_j - w_i \geq s_i - w_j$

And from (2), $-W \geq w_j - s_i$

So $s_j - w_i - W \geq (s_i - w_j) + (w_j - s_i) = 0$, as required.

(v) $w_j - s_i \geq w_i - s_j$ is equivalent to $w_j + s_j \geq w_i + s_i$

So, from (iv), an order is not made worse (ie changing from a safe order to an unsafe one) by swapping rows so that the higher value of $w_r + s_r$ is moved to the lower row.

We can therefore start with the rows ordered by the size of $w_r + s_r$, with the smallest value at the top (order X say), and this cannot be worse than any other order.

If a safe order exists then order X will be one of them (it may be the only one though, as in the case of the fruit example).