2017 MAT Paper - Multiple Choice (7 pages; 28/8/20)

Q1/A

Solution

$f(x) = 2x^3 - kx^2 + 2x - k$

$\Rightarrow f'(x) = 6x^2 - 2kx + 2$

There will be two distinct stationary points when $\Delta > 0$;

ie when $(-2k)^2 - 4(6)(2) > 0$

$\Leftrightarrow k^2 > 12 \Leftrightarrow k < -\sqrt{12} = -2\sqrt{3}$ or $k > 2\sqrt{3}$

So the answer is (b).

Note: In the general case, where $f(x) = \mathrm{a}x^3 + \mathrm{b}x^2 + \mathrm{c}x + d$,

a cubic function has two distinct stationary points when $b^2 - 3ac$. (Also, the point of inflexion (about which there is rotational symmetry) is at $\frac{-b}{3a}$.)

Q1/B

Solution

Writing $y = cos^2x$, $f(y) = 9y^2 - 12y + 7$

This has its minimum value when $y = -\frac{b}{2a} = \frac{12}{18} = \frac{2}{3}$

and $f(y) = 9(\frac{2}{3})^2 - 12\left(\frac{2}{3}\right) + 7 = 4 - 8 + 7 = 3$

So the answer is (a).

(Alternatively, we can complete the square.)

Q1/C

Introduction

A large number, such as 2017, suggests that the sequence may well be periodic.

Solution

$a_3 = \frac{a_2}{a_1} = \frac{6}{2} = 3$, $a_4 = \frac{3}{6} = \frac{1}{2}$, $a_5 = \frac{(\frac{1}{2})}{3} = \frac{1}{6}$, $a_6 = \frac{(\frac{1}{6})}{(\frac{1}{2})} = \frac{1}{3}$

$a_7 = \frac{(\frac{1}{3})}{(\frac{1}{6})} = 2$, $a_8 = \frac{2}{(\frac{1}{3})} = 6$

As $a_7 = a_1$ and $a_8 = a_2$, subsequent terms will repeat the ones obtained so far, with the sequence having period 6.

As $2017 = 6 \times 336 + 1$, $a_{2017} = a_1 = 2$

So the answer is (d).

Q1/D

Solution

$f(x) \to f(-x)$ is a reflection in the y-axis, and $f(x) \to -f(x)$ is a reflection in the x-axis, so $f(x) \to -f(-x)$ is a rotation through 180°, and hence **the answer is (c)**

Q1/E

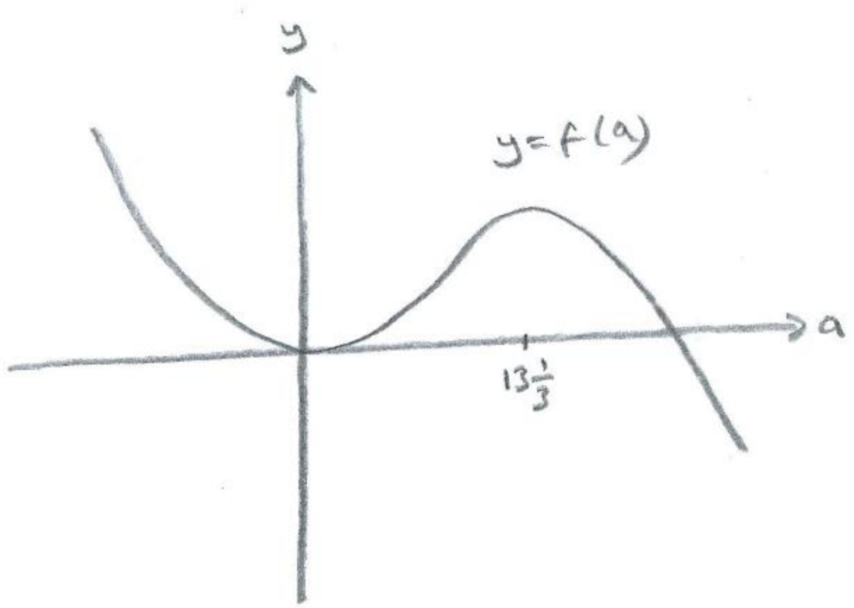
Solution

$f(a) = a^2 b = a^2(20 - a) = 20a^2 - a^3$

$f'(a) = 0 \Rightarrow 40a - 3a^2 = 0 \Rightarrow a = 0$ (reject) or $a = \frac{40}{3} = 13\frac{1}{3}$

From the shape of the cubic $y = f(a)$ (see below), the maximum value of $f(a)$ for positive integer a will be at either $a = 13$

or $a = 14$.



$f(13) = 169(7) = 1183$, whilst $f(14) = 196(6) = 1176$

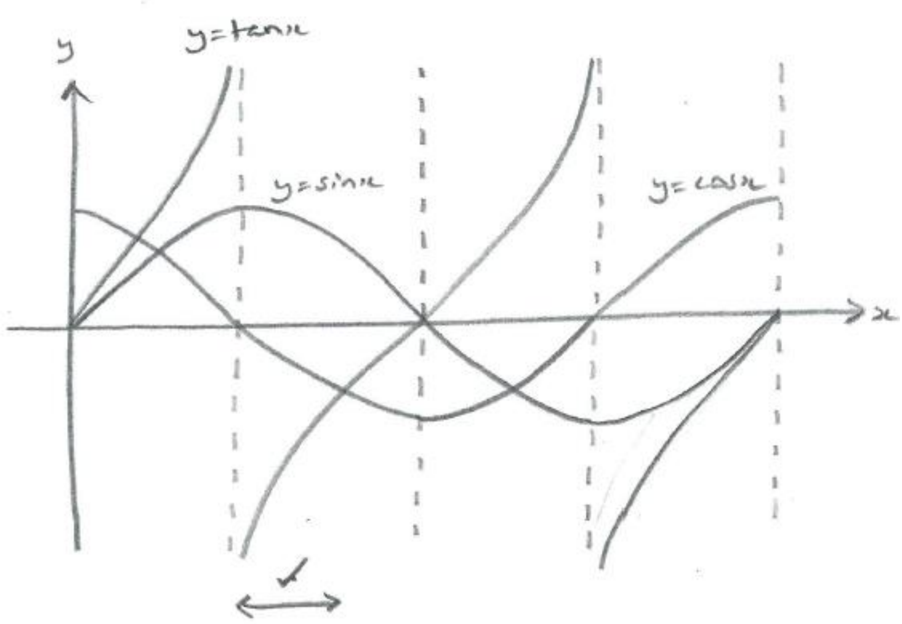
and so **the answer is (d)**

Note: the official answer says that 13 is the closest integer to $13\frac{1}{3}$, and so 13 is the required value of a. However, this isn't in itself a reason for $f(a)$ being maximised at $a = 13$.

Q1/F

[The labelling of the axes implies the use of parametric equations. More usually, the horizontal variable would be $x = cos\theta$ and the vertical variable would be $y = sin\theta$, where θ is the angle that OP makes with the positive x-axis - where P is the point (x, y) on the unit circle.]

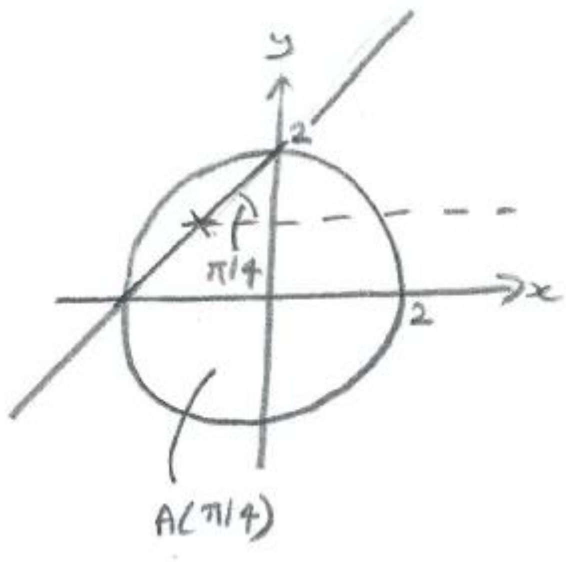
Solution



By considering the graphs of $tanx$, $cosx$ & $sinx$, it can be seen that **the answer is (c).**

Q1/G

Solution



The given line can be written as $\frac{y-1}{x-(-1)} = tan\theta$, and so it passes through the point $(-1, 1)$, at an angle θ to the x-axis.

Consider first of all the area $A(0)$, when the line is parallel to the x-axis. As θ starts to increase, so that the line rotates about $(-1, 1)$, it can be seen that the larger region gains from the smaller region, and so $A(\theta)$ increases. This continues until $\theta = \frac{\pi}{4}$, when the process starts to be reversed. $A(\theta)$ is maximised again when $\theta = \frac{\pi}{4} + \pi$. Thus there are two values of θ in the range

$0 \leq \theta < 2\pi$ for which $A(\theta)$ is maximised.

So the answer is (b).

Q1/H

Solution

By the Remainder/Factor theorems,

$(-b)^2 - 2a(-b) + a^4 = 1$ and $b\left(\frac{1}{a}\right)^2 + \frac{1}{a} + 1 = 0;$

ie $b^2 + 2ab + a^4 = 1$ (1) and $b + a + a^2 = 0$ (2)

[Attempting to eliminate a:]

(1) can be written as $(b + a)^2 - a^2 + a^4 = 1$

Then, from (2), $a^4 - a^2 + a^4 = 1$

$\Rightarrow a^4 - a^2 = 1 - a^4$

$\Rightarrow a^2(a^2 - 1) = (1 - a^2)(1 + a^2)$

$\Rightarrow a^2 = 1 \; or \; a^2 = -(1 + a^2)$

As $-(1 + a^2) < 0$, the 2nd option isn't possible.

If $a = 1$, $(1) \Rightarrow b^2 + 2b = 0$ and $(2) \Rightarrow b = -2$

As these are consistent, one solution is $b = -2$

If $a = -1$, $(1) \Rightarrow b^2 - 2b = 0$ and $(2) \Rightarrow b = 0$

As these are consistent, another solution is $b = 0$

So the answer is (b).

Q1/I

Solution

$log_b((b^x)^x) = x^2$ and $log_a\left(\frac{c^x}{b^x}\right) = xlog_a\left(\frac{c}{b}\right)$

So the equation is a quadratic in x and has a repeated root when

$[log_a\left(\frac{c}{b}\right)]^2 - 4log_a\left(\frac{1}{b}\right)log_a(c) = 0$

$\Rightarrow (log_a c - log_a b)^2 + 4log_a b log_a c = 0$

$\Rightarrow (log_a c + log_a b)^2 = 0$

$\Rightarrow log_a c = -log_a b = log_a(\frac{1}{b})$

$\Rightarrow c = \frac{1}{b}$

So the answer is (d).

Q1/J

Introduction

A common approach for this type of question is to show, for example, that $A \leq a$ and $B \geq b$, where $a < b$, so that $A < B$

(or $A < a$ and $B \geq b$, where $a \leq b$ etc).

So, for this question, we could try to find a relation of the form $A \leq a$ in 4 of the cases, and one relation of the form $B \geq b$ (or something similar).

Solution

We can see that (a) < 0.

(b): The maximum value of $(2 + cosx)^3$ occurs when $cosx = 1$, and is 27. So (b) $< 2\pi(27) = 54\pi$.

(c) $< \pi(1) = \pi$

(d): The minimum value of $(3 - sinx)^6$ occurs when $sinx = 1$, and is $2^6 = 64$. So (d) $> \pi(64) = 64\pi$.

(e) As $(sin^3x - 1) \leq 0$, (e) < 0

So the answer is (d), as $64\pi > 54\pi$.