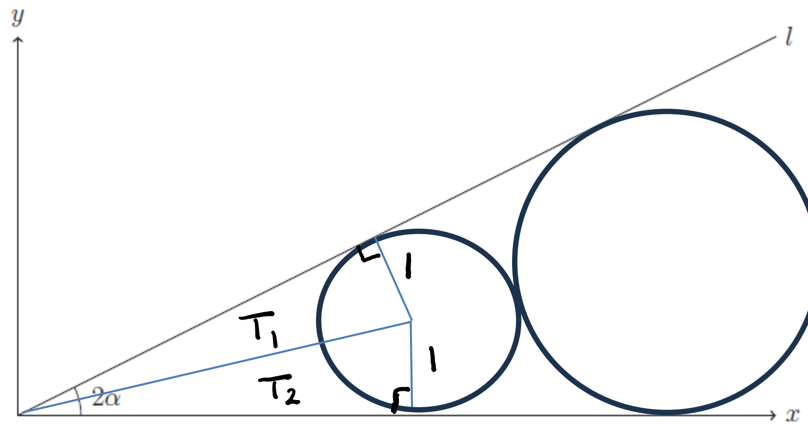
2016 MAT – Q4 (3 pages;18/11/23)

Solution

(i)



(not to scale)

The triangles T_1 and T_2 are both right-angled, have a side of length 1, and share a common hypotenuse. They are therefore congruent, and so both have an angle of α at O.

Hence the base of T_2 is $\frac{1}{tan\alpha}$, and so the centre of circle C_1 is

$(cot\alpha, 1)$.

(ii) The equation of C_1 is therefore $(x - cot\alpha)^2 + (y - 1)^2 = 1$

[Unusually, this part doesn't seem to be relevant to the rest of the question.]

(iii) Similarly, the centre of circle C_2 is $(3cot\alpha, 3)$.

When the two circles are touching, the length of the line segment joining the centres of the two circles is $1 + 3 = 4$ (as the common tangent to the two circles is perpendicular to the radii of the two circles at their point of contact).

By Pythagoras' theorem, $(3cot\alpha - cot\alpha)^2 + (3-1)^2 = 4^2$,

so that $4cot^2\alpha + 4 = 16$,

and hence $4cosec^2\alpha = 16$,

so that $sin^2\alpha = \frac{1}{4}$,

and therefore $sin\alpha = \frac{1}{2}$ and $\alpha = \frac{\pi}{6}$ (given that $\alpha < \frac{\pi}{4}$)

(iv) Let the radius of C_3 be r.

Applying the same method as in (ii),

$(rcot\alpha - 3cot\alpha)^2 + (r-3)^2 = (3+r)^2$,

giving $(r-3)^2(cot^2\alpha + 1) = (3+r)^2$,

Then, as $cot^2\alpha + 1 = cosec^2\alpha = \frac{1}{sin^2\alpha} = \frac{1}{(\frac{1}{4})} = 4$,

we have $3r^2 - 30r + 27 = 0$,

or $r^2 - 10r + 9 = 0$,

so that $(r-9)(r-1) = 0$, and hence $r = 9$ ($r = 1$ giving C_1, and C_3 is to be larger than C_2)

Thus the radius of C_3 is 9.

[This result could just have been obtained by noting that the ratio of the radii of C_3 and C_2 must be the same as that of the radii of C_2 and C_1. (We can consider a change of scale, writing $x' = 3x$, and apply this to C_1 and C_2.)]

(v) The area of triangle T_1 is $\frac{1}{2}\cot\left(\frac{\pi}{6}\right)(1) = \frac{1}{2\tan\left(\frac{\pi}{6}\right)} = \frac{\sqrt{3}}{2}$

Consider the triangle $T_1{}'$, which bears the same relation to

C_2 that T_1 bears to C_1.

Then the area of triangle $T_1{}'$ is $\frac{1}{2}(3\cot\left(\frac{\pi}{6}\right))(3) = \frac{9\sqrt{3}}{2}$

The required area is then equal to $\frac{9\sqrt{3}}{2} - \frac{\sqrt{3}}{2}$, less the relevant sectors of C_1 and C_2.

The angles of these sectors are

$\pi - \left(\frac{\pi}{2} - \alpha\right) = \frac{\pi}{2} + \frac{\pi}{6} = \frac{2\pi}{3}$ for C_1

and $\frac{\pi}{2} - \alpha = \frac{\pi}{3}$ for C_2

So the required area is $4\sqrt{3} - \pi(1)^2 \times \frac{\left(\frac{2\pi}{3}\right)}{2\pi} - \pi(3)^2 \times \frac{\left(\frac{\pi}{3}\right)}{2\pi}$

$= 4\sqrt{3} - \pi(\frac{1}{3} + \frac{3}{2})$

$= 4\sqrt{3} - \frac{11\pi}{6}$ sq. units